1. Let us consider the action describing a point particle of mass m moving freely in a $d+1$ dimensional Minkowski space–time. It will be expressed as the invariant length of the world–line

$$S = \int_{\tau_0}^{\tau_1} d\tau\, \mathcal{L} = -m \int_{\tau_0}^{\tau_1} ds = -m \int_{\tau_0}^{\tau_1} d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}. \tag{1}$$

where $x^\mu(\tau)$ is the space–time position of the particle at the proper time τ and $\dot{x}^\mu = dx^\mu/d\tau$. The space–time metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

- Show that S is invariant with respect to *reparametrization* of the world line: $\tau \to \tau' = \tau'(\tau)$.
- Compute the momentum of the particle $p^\mu = \eta^{\mu\nu}\, \delta\mathcal{L}/\delta\dot{x}^\nu$ and show that it describes a particle of mass m.
- Find the equation of motion for $x^\mu(\tau)$ by minimizing the action with respect to a variation $x^\mu \to x^\mu + \delta x^\mu$ and find the most general solution. In particular show that, if we give a physical interpretation to τ as being the time, namely if we set $x^0(\tau) \propto \tau$, the solution has $v^\mu = \dot{x}^\mu = \mathrm{const}$. Show that on the solutions the action can be written in the form:

$$S = -m\, |x_1 - x_0|. \tag{2}$$

$$\begin{aligned} |x_1 - x_0| &= \sqrt{-(x_1^\mu - x_0^\mu)(x_{1\mu} - x_{0\mu})} \\ x_0^\mu &= x^\mu(\tau_0)\ ;\ \ x_1^\mu = x^\mu(\tau_1) \\ x_1^\mu - x_0^\mu &= v^\mu\, (\tau_1 - \tau_0). \end{aligned} \tag{3}$$

and that the momentum can be written as:

$$p^\mu = m\, \frac{(x_1^\mu - x_0^\mu)}{|x_1 - x_0|}. \tag{4}$$

(for later convenience we allow p^0 to have also negative values and we interpret the energy of the particle to be $E = |p^0|$. According to this notation, in a scattering process, the incoming particles will have $p^0 = E > 0$ while the outgoing will have $p^0 = -E < 0$.)

- Now consider a different action on the same world line of the particle, but this time function of two independent quantities, namely $e(\tau)$ and $x^\mu(\tau)$:

$$S' = \frac{1}{2}\int (e^{-1}\,\dot{x}^\mu \dot{x}_\mu - e m^2) \tag{5}$$

Show that S' is invariant with respect to the reparametrization transformation in τ which, in its infinitesimal form is:

$$\begin{aligned} \delta x^\mu &= \epsilon(\tau)\,\dot{x}^\mu \\ \delta e &= \frac{d}{d\tau}\,(\epsilon(\tau)\,e) \\ \epsilon(\tau_0) &= \epsilon(\tau_1) \,=\, 0. \end{aligned} \tag{6}$$

(**hint:** *to show invariance of the action under the above infinitesimal transformations it suffices to show that the corresponding variation of the action is a total derivative of the form written below:*

$$\mathcal{L}(e+\delta e, \dot{x}^\mu + \delta\dot{x}^\mu) = \mathcal{L}(e, \dot{x}^\mu) + \frac{d}{d\tau}\,(\epsilon\mathcal{L})\,. \tag{7}$$

for these transformations the above result follows without the use of the equations of motion. Since by hypothesis ϵ vanishes at the extrema of integration the total derivative in eq. (7) does not contribute.) Compute the field equations from S' corresponding to the fields $e(\tau)$ and $x^\mu(\tau)$ and show that S' is classically equivalent to S.

- Consider a scattering process of N particles described by the world lines of the particles which start at different points x_i^μ $(i = 1, \ldots, N)$ and intersect in the same scattering point y^μ. Let the process be described by the total action $S = \sum_{i=1}^N S_i$, S_i being the world line actions of the single particles, by minimizing S with respect to the position y^μ of the scattering point deduce the momentum conservation condition for the process: $\sum_{i=1}^N p_i^\mu = 0$ (recall that in our notation the incoming particles have $p^0 > 0$ and the outgoing have $p^0 < 0$).
hint: *use eq. (2) to express the single actions as $S_i = -m_i\,|y - x_i|$. Then from the condition $\delta S/\delta y^\mu = 0$ and the expression (4) for the momenta deduce $\sum_{i=1}^N p_i^\mu = 0$.*

2. A generic surface $\mathcal{S}$ spanned by affine parameters τ, σ and embedded in a higher dimensional space (ambient space) with coordinates X^μ, can be described by the parametric equations $X^\mu = X^\mu(\tau, \sigma)$. An infinitesimal element of its surface $d\mathcal{S}$ can be represented by a tensor $d\mathcal{S}^{\mu\nu}(\tau, \sigma) = (dX^\mu \wedge dX^\nu)_{|\mathcal{S}} = \Sigma^{\mu\nu}(\tau, \sigma)d\tau d\sigma$ which defines the projection of $d\mathcal{S}$ on the plane (X^μ, X^ν), the tensor $\Sigma^{\mu\nu}(\tau, \sigma)$ defines the plane tangent to the surface $\mathcal{S}$ at the point (τ, σ) and is defined as follows:
$$\Sigma^{\mu\nu} = \partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu \tag{8}$$
. The area of the surface $A(\mathcal{S})$ is then defined by the integral:
$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{\frac{1}{2} d\mathcal{S}_{\mu\nu} d\mathcal{S}^{\mu\nu}} = \int_{\mathcal{S}} d\tau d\sigma \sqrt{\frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu}} \tag{9}$$
Compute the Σ tensor and the area of the surface given by:
$$\begin{aligned} X^0 &= A\tau \\ X^1 &= -B\tau \\ X^2 &= \sigma \end{aligned}$$
where A and B are constants, $\tau \in (0, 1)$ and $\sigma \in (0, 1)$. You may consider the metric of the ambient space to be Euclidean. Compare the result of the integral with what you would expect.

3. Consider the Nambu–Goto action described by equation (2.7) of the lecture notes. Show that the expression in the square root can be written as the determinant of a 2×2 matrix (induced metric) $h_{\alpha\beta}$ defined as
$$h_{\alpha\beta} = \partial_\alpha X^\mu \, \partial_\beta X_\mu \tag{10}$$
In the Nambu–Goto action the only independent function is $X^\mu(\sigma, \tau)$. It is possible to reformulate the theory in a classically equivalent way using the Polyakov action which describes $h_{\alpha\beta}(\sigma, \tau)$ and $X^\mu(\sigma, \tau)$ as independent fields and has the advantage of not having the square root in the integral (see next exercise)

4. Consider the Polyakov action of a string moving on a D–dimensional Minkowski background (with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$):

$$S = -\frac{T}{2} \int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, h^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu\, \partial_\beta X_\mu \tag{11}$$

$$\begin{aligned}
\sigma^\alpha &= \{\sigma, \tau\} \ ; \ X^\mu = X^\mu(\sigma^\alpha)\, ; \, \sigma \in [0, \pi]\, ; \, \tau \in (-\infty, \infty) \\
\Sigma &= \text{world sheet} \ ; \ h_{\alpha\beta}(\sigma^\gamma) = \text{metric on } \Sigma \\
h(\sigma^\gamma) &= -\det(h_{\alpha\beta}) \ ; \ \alpha = 1, 2\, ; \, \mu = 0, \ldots, \mathrm{D}-1
\end{aligned}$$

where we have used the following short hand notation for partial derivatives: $\partial_\gamma = \frac{\partial}{\partial \sigma^\gamma}$. Moreover whenever repeated upper and lower indices occur summation is understood: $v^\alpha w_\alpha = \sum_\alpha v^\alpha w_\alpha$.

- Show that the *local*, i.e. σ^α–dependent, symmetry transformations are:

$$\begin{aligned}
\underline{\text{reparametrization}}: &\qquad \sigma^\alpha \to \sigma^{\alpha\prime} = \sigma^{\alpha\prime}(\sigma) &(12)\\
\underline{\text{Weyl transformations}}: &\qquad h_{\alpha\beta} \to \Omega^2(\sigma^\alpha)\, h_{\alpha\beta} &(13)
\end{aligned}$$

- Compute the energy momentum tensor $T_{\alpha\beta}$ defined by:

$$\begin{aligned}
h_{\alpha\beta} &\to h_{\alpha\beta} + \delta h_{\alpha\beta} \Rightarrow S \to S + \delta S \\
\delta S &= -\frac{T}{2} \int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, \delta h_{\alpha\beta}\, T^{\alpha\beta} &(14)
\end{aligned}$$

  which condition on $T_{\alpha\beta}$ does invariance under Weyl transformations imply?

- Show that the *global*, i.e. σ^α–independent, transformations on the X^μ fields which leave S invariant are the Poincaré transformations:

$$\begin{aligned}
X^\mu &\to X'^\mu = \Lambda^\mu{}_\nu\, X^\nu + a^\mu \\
\eta_{\mu\nu}\, \Lambda^\nu{}_\rho\, \Lambda^\mu{}_\gamma &= \eta_{\rho\gamma} &(15)
\end{aligned}$$

  where both the Lorentz transformation Λ and the translation parameter a^μ do not depend on σ^α.

- Write the equations of motion for the fields $h_{\alpha\beta}(\sigma)$ and $X^\mu(\sigma)$ and show that S is *classically* equivalent to the Nambu–Goto action S':

$$S' = T \int_\Sigma d\sigma^2 \sqrt{-\det\left(\partial_\alpha X^\mu\, \partial_\beta X_\mu\right)} \tag{16}$$

5. By fixing reparametrization invariance let us reduce the world sheet metric to the form:

$$h_{\alpha\beta}(\sigma) \;=\; \lambda(\sigma)\,\eta_{\alpha\beta} \;=\; \lambda(\sigma)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

- write the action S with this metric
- consider an infinitesimal coordinate transformation on the world sheet:

$$\sigma^\alpha \;\to\; \sigma^{\alpha\prime} \;=\; \sigma^\alpha + \epsilon^\alpha(\sigma) \tag{18}$$

  which implies that we transform X^μ as follows:

$$X^\mu \;\to\; X^\mu + \delta X^\mu \;=\; X^\mu + \epsilon^\gamma\,\partial_\gamma X^\mu$$

  As it can be deduced from eq. (10), the metric $h_{\alpha\beta}$ will transform as follows:

$$\delta h_{\alpha\beta} \;=\; (\partial_\alpha \epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta \epsilon^\gamma)\,h_{\gamma\alpha} + \epsilon^\gamma \partial_\gamma h_{\alpha\beta} \tag{19}$$

  After fixing the metric to the form (17) there is still a residual invariance of the action under *conformal* transformations. A conformal transformation is defined as a coordinate transformation whose only effect is to rescale the metric, namely such that the corresponding infinitesimal variation of the metric has the general form:

$$\delta h_{\alpha\beta} \;=\; C(\sigma)\,h_{\alpha\beta} + \epsilon^\gamma \partial_\gamma h_{\alpha\beta} \tag{20}$$

  where $C(\sigma)$ is an infinitesimal function of σ^α which depends on the infinitesimal coordinate shift ϵ. Using equ. (19) show that the conformal transformations are generated by an infinitesimal parameter ϵ fulfilling the following condition:

$$(\partial_\alpha \epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta \epsilon^\gamma)\,h_{\gamma\alpha} \;=\; (\partial_\gamma\,\epsilon^\gamma)\;h_{\alpha\beta} \tag{21}$$

  recall the convention on repeated indices and partial derivation: $(\partial_\gamma\,\epsilon^\gamma = \sum_\gamma \frac{\partial}{\partial\sigma^\gamma}\,\epsilon^\gamma)$ and find the expression of $C(\sigma)$ in equ. (20) in terms of ϵ.

- using the light–cone coordinates:

$$\sigma^{\pm} \; = \; \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{22}$$

show that conformal transformations are characterized by $\epsilon^{+} = \epsilon^{+}(\sigma^{+})$ and $\epsilon^{-} = \epsilon^{-}(\sigma^{-})$.

- Write $T_{\alpha\beta}$ and the conditions on it due to energy–momentum conservation and Weyl invariance, in light–cone coordinates.

6. Consider the global symmetries of S, i.e. the Poincaré transformations in their infinitesimal form:

$$\begin{aligned}
X^{\mu} &\rightarrow X^{\mu} + \delta X^{\mu} && (23)\\
\delta X^{\mu} &= a^{\mu} \qquad \text{(translations)} && (24)\\
\delta X^{\rho} &= \omega_{\mu\nu}\,(M^{\mu\nu})^{\rho}{}_{\sigma}\,X^{\sigma} \quad \text{(Lorentz)} && (25)\\
(M^{\mu\nu})^{\rho}{}_{\sigma} &= \eta^{\nu\rho}\,\delta^{\mu}_{\sigma} - \eta^{\mu\rho}\,\delta^{\nu}_{\sigma} &&\\
\eta^{\mu\nu} &= \mathrm{diag}(-1,+1,\ldots,+1) && (26)
\end{aligned}$$

where a^{μ} and $\omega_{\mu\nu}$ are σ^{α}–independent.

- Compute the corresponding conserved Noether currents J^{μ}_{α} and $J^{\mu\nu}_{\alpha}$ by computing the variation of the Polyakov action S with respect to the transformations (24) and (24) and expressing it (using the field equations) in the form:

$$\delta S \; = \; \int d\sigma d\tau\, \partial^{\alpha}\,(J^{\mu}_{\alpha}\,a_{\mu} + J^{\mu\nu}_{\alpha}\,\omega_{\mu\nu}) \tag{27}$$

recall that we are using here and in all the following exercises $h_{\alpha\beta} = \eta_{\alpha\beta}$

- Write the equations of motion for X^{μ} and solve them with the following boundary conditions:

$$\begin{aligned}
&\underline{\text{Neumann:}} && \partial_{\sigma}X^{\mu}(\tau,\sigma=0) \;=\; \partial_{\sigma}X^{\mu}(\tau,\sigma=\pi) \;=\; 0\\
&\underline{\text{Closed string:}} && X^{\mu}(\tau,\sigma) \;\equiv\; X^{\mu}(\tau,\sigma+\pi) \quad \forall\sigma
\end{aligned}$$

showing that the most general solution will have the form:

$$X^{\mu}(\sigma) \; = \; X^{\mu}_{L}(\sigma^{+}) + X^{\mu}_{R}(\sigma^{-}) \tag{28}$$

(express solution through Fourier mode expansion and impose the boundary conditions as constraints on the coefficients)

- Show that for the above solutions the CM momentum:

$$P^\mu = \int_0^\pi d\sigma \, J^\mu_\tau \tag{29}$$

is conserved.

- Write the expression of P^μ for the above solutions.
- Consider along a certain direction $X^{\tilde{\mu}}$ a solution with Dirichlet boundary conditions:

$$\partial_\tau X^{\tilde{\mu}}(\tau, \sigma = 0) = \partial_\tau X^{\tilde{\mu}}(\tau, \sigma = \pi) = 0 \tag{30}$$

is the momentum along the same direction $P^{\tilde{\mu}}$ conserved?

7. Show that the following functions:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\sin(\tau)\cos(\sigma) \\ X^{i>2} &= 0 \end{aligned} \tag{31}$$

define a solution of the string field equations with the Neumann boundary conditions (i.e. describe a free open string). In particular show that it can be written in the form (28).

- Compute the energy $E = P^0$ and the angular momentum $\vec{J}$ for this solution.
- Show that the non linear constraints $T_{\alpha\beta} = 0$ are fulfilled as well, namely that :

$$\begin{aligned} (\partial_\tau X)^2 + (\partial_\sigma X)^2 &= 0 \\ \partial_\tau X^\mu \, \partial_\sigma X_\mu &= 0 \end{aligned} \tag{32}$$

- Show that:

$$\frac{E^2}{|\vec{J}|} = \text{const.} = \frac{1}{\alpha'} \tag{33}$$

and show that $T = 1/(2\pi\alpha')$

- Show that this solution describes an open string with the end points rotating at the speed of light.

8. Consider the following parametric equations:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\cos(\tau)\sin(\sigma) \end{aligned} \tag{34}$$

Show that they describe a string solution, i.e. that one can write it in the form (28) and that the non linear constraints in eq. (32) are fulfilled. Which boundary conditions does the solution fulfill in the various space directions?

- Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to 2π with steps of $\pi/4$.
- Compute the conserved current J^μ_α associated with global translational invariance in space–time (see problem 6). Show that the component $P^{\mu=2}$ of the momentum defined by eq. (29) is not conserved.
- Compute the variation of the momentum between $\tau = 0$ and $\tau = \tau_0$ and prove the following relation:

$$P^{\mu=2}_{|\tau=\tau_0} - P^{\mu=2}_{|\tau=0} = \int_0^{\tau_0} d\tau \, \left(J^{\mu=2}_\sigma(\tau,\pi) - J^{\mu=2}_\sigma(\tau,0) \right) \tag{35}$$

  the above equation derives from the momentum conservation condition $\partial^\alpha J^\mu_\alpha = 0$ and the right hand side can be interpreted as the momentum flow in the direction $\mu = 2$ across the end–points of the string.

9. Consider the following parametric equations:

$$\begin{aligned} X^0 &= 2A\tau \\ X^1 &= A\cos 2\tau\cos 2\sigma \\ X^2 &= A\sin 2\tau\cos 2\sigma \end{aligned} \tag{36}$$

Show that they describe a closed string solution and that one can write it in the form (28) and that it fulfills (32). Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to π with steps of $\pi/8$.

10. Consider the following parametric equations:

$$
\begin{aligned}
X^0 &= A\tau \\
X^1 &= A\cos(2\tau)\sin(2\sigma) \\
X^2 &= A\sin(4\tau)\sin(4\sigma)
\end{aligned}
\tag{37}
$$

Show that although they can be written in the form (28) they do not fulfill the constraint (32).

11. Write the components T_{++} and T_{--} of the energy momentum tensor in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions. In particular in the two cases find the corresponding expressions of the coefficients L_n and $\tilde{L}_n$ defined as follows:

$$
\begin{aligned}
&\underline{\text{open string:}} && T_{++} = \frac{\ell^2}{2}\sum_{n=-\infty}^{\infty} L_n\, e^{-in\sigma^+} \\
&\underline{\text{closed string:}} && T_{++} = 2\ell^2\sum_{n=-\infty}^{\infty} \tilde{L}_n\, e^{-2in\sigma^+} \\
& && T_{--} = 2\ell^2\sum_{n=-\infty}^{\infty} L_n\, e^{-2in\sigma^-}
\end{aligned}
\tag{38}
$$

The constraints $T_{\alpha\beta} = 0$ can be restated as $L_n = \tilde{L}_n = 0$ for all n. From the conditions $L_0 = \tilde{L}_0 = 0$ derive an expression for the mass squared $M^2 = -p^\mu p_\mu$ in terms of the coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ in both the open and closed string cases.

12. Compute in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions the Hamiltonian associated with the Polyakov action:

$$
H(\tau) = \int d\sigma \left(\dot{X}^\mu \frac{dP_\mu}{d\sigma}\right) - L(\tau) \tag{39}
$$

where $\frac{dP_\mu}{d\sigma}$ and $L(\tau)$ are defined in eqs. (3.38) and (4.1) of the lecture notes. What is the relation between H and L_0, $\tilde{L}_0$ in the open and closed string cases?

13. Let us restrict ourselves to the quantized open string case in the light-cone gauge. Derive the commutation relations between x^i, p^i and α^i_n from equations (4.5) and (4.3) of the lecture notes. Let us define the vacuum state $|p^\mu, 0\rangle$ such that $\alpha^i_n |p^\mu, 0\rangle = 0$ for $n < 0$ and $\alpha^i_0 |p^\mu, 0\rangle = p^i |p^\mu, 0\rangle$. A generic open string state is obtained by applying the creation operators α^i_n $(n < 0)$ a finite number of times to $|p^\mu, 0\rangle$. Consider the operator N defined as follows:

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha^i_{-n} \alpha^i_n \tag{40}$$

Show that on the following state:

$$|p^\mu,\, N_{i,n}\rangle = \overbrace{\alpha^i_{-n} \ldots \alpha^i_{-n}}^{N_{i,n}} |p^\mu, 0\rangle \tag{41}$$

the operator N has the following value:

$$N\, |p^\mu,\, N_{i,n}\rangle = n N_{i,n} |p^\mu,\, N_{i,n}\rangle \tag{42}$$

14. A solution of the string equations of motion can be written in the form (28). For the open string we have the further constraint that $X^\mu_L(\tau) = X^\mu_R(\tau)$ and that $X^\mu_L(\sigma^+)$ and $X^\mu_R(\sigma^-)$ are respectively periodic in σ^+ and σ^- of period 2π a part for a constant shift while for the closed string X^μ_L and X^μ_R need only be periodic in $\sigma \in (0, \pi)$ at fixed τ. The theory is also invariant under conformal transformations which can be written in the following infinitesimal form:

$$\begin{aligned} \sigma^+ &\to \sigma^+ + \epsilon^+(\sigma^+) \\ \sigma^- &\to \sigma^- + \epsilon^-(\sigma^-) \end{aligned} \tag{43}$$

where $\epsilon^+(\sigma^+)$ and $\epsilon^-(\sigma^-)$ are periodic in σ^+ and σ^- of period 2π. For the open string we further require that $\epsilon^+ = \epsilon^-$. In this exercise we will compute the infinitesimal generators of conformal transformations and their algebraic properties. Being periodic of period 2π in their argument, the infinitesimal functions can be expanded in Fourier series:

$$\epsilon^\pm(\sigma^\pm) = i \sum_{n=-\infty}^{+\infty} \epsilon^\pm_n e^{in\sigma^\pm} \tag{44}$$

where the reality condition implies that $\epsilon^{\pm}_{-n} = -(\epsilon^{\pm}_n)^{\star}$.

A generic scalar function $Y(\sigma^{\pm})$ will transform under (43) as follows:

$$
\begin{aligned}
Y(\sigma^{\pm}) &\rightarrow Y(\sigma^{\pm} + \epsilon^{\pm}) \sim Y(\sigma^{\pm}) + \delta Y(\sigma^{\pm}) \\
\delta Y(\sigma^{\pm}) &= \epsilon^{\pm}\partial_{\pm} Y(\sigma^{\pm}) = i \sum_{n=-\infty}^{+\infty} \epsilon^{\pm}_n e^{in\sigma^{\pm}} \partial_{\pm} Y(\sigma^{\pm})
\end{aligned} \tag{45}
$$

where we have used eq. (44) to derive the last expression. Let us define the following differential operators $\mathbf{L}^{(\pm)}_n$ on functions of $\sigma^{\pm}$ respectively:

$$
\begin{aligned}
\mathbf{L}^{(+)}_n \Big(Y(\sigma^{+})\Big) &= ie^{in\sigma^{+}} \partial_{+} Y(\sigma^{+}) \\
\mathbf{L}^{(-)}_n \Big(Y(\sigma^{-})\Big) &= ie^{in\sigma^{-}} \partial_{-} Y(\sigma^{-}) \\
\mathbf{L}^{(+)}_n \Big(Y(\sigma^{-})\Big) &= 0 \\
\mathbf{L}^{(-)}_n \Big(Y(\sigma^{+})\Big) &= 0
\end{aligned} \tag{46}
$$

the product of two of these operators is defined by their consecutive action on a same function Y: $\mathbf{L}_n \, \mathbf{L}_m \, (Y) \equiv \mathbf{L}_n \, (\mathbf{L}_m \, (Y))$. The operators $\mathbf{L}^{(\pm)}_n$ are the generators of the infinitesimal conformal transformations (45), indeed we can write:

$$
\delta Y(\sigma^{\pm}) = \sum_{n=-\infty}^{+\infty} \epsilon^{\pm}_n \, \mathbf{L}^{(\pm)}_n \, (Y) \tag{47}
$$

- Show that taking $Y(\sigma^{\pm}) = \sigma^{\pm}$ and considering the definitions (46) the infinitesimal conformal transformation (43) follows from (47).
- Show that on a generic function $Y(\sigma^{\pm})$ the following relations hold:

$$
\begin{aligned}
\Big(\mathbf{L}^{(+)}_n \, \mathbf{L}^{(+)}_m - \mathbf{L}^{(+)}_m \, \mathbf{L}^{(+)}_n\Big) \Big(Y(\sigma^{+})\Big) &= (n-m)\mathbf{L}^{(+)}_{n+m} \Big(Y(\sigma^{+})\Big) \\
\Big(\mathbf{L}^{(-)}_n \, \mathbf{L}^{(-)}_m - \mathbf{L}^{(-)}_m \, \mathbf{L}^{(-)}_n\Big) \Big(Y(\sigma^{-})\Big) &= (n-m)\mathbf{L}^{(-)}_{n+m} \Big(Y(\sigma^{-})\Big) \\
\Big(\mathbf{L}^{(+)}_n \, \mathbf{L}^{(-)}_m - \mathbf{L}^{(-)}_m \, \mathbf{L}^{(+)}_n\Big) \Big(Y(\sigma^{\pm})\Big) &= 0
\end{aligned} \tag{48}
$$

- Consider the following two constant infinitesimal transformations:

$$
\begin{aligned}
\delta\tau &= c \; ; \; \delta\sigma = 0 \qquad \text{time shift} \\
\delta\tau &= 0 \; ; \; \delta\sigma = c \qquad \text{world sheet rotation}
\end{aligned} \tag{49}
$$

Show that they are generated respectively by the following differential operators:

$$\begin{aligned}\mathcal{O}_t &= -i\,c\,(\mathbf{L}_0^{(+)} + \mathbf{L}_0^{(-)}) \\ \mathcal{O}_s &= -i\,c\,(\mathbf{L}_0^{(+)} - \mathbf{L}_0^{(-)})\end{aligned} \tag{50}$$

namely that for the time shift $\delta\tau = \mathcal{O}_t(\tau) = c$ and $\delta\sigma = \mathcal{O}_s(\sigma) = 0$ while for the world sheet rotation $\delta\tau = \mathcal{O}_s(\tau) = 0$ and $\delta\sigma = \mathcal{O}_s(\sigma) = c$

15. Let us define the following operators on the closed string states:

$$\begin{aligned}\mathcal{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\alpha^i_m \alpha^i_{n-m}: \\ \tilde{\mathcal{L}}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\tilde{\alpha}^i_m \tilde{\alpha}^i_{n-m}:\end{aligned} \tag{51}$$

where as usual α_n and $\tilde{\alpha}_n$ denote the right and left moving mode operators. Show that they fulfill the following commutation relations:

$$\begin{aligned}\left[\mathcal{L}_n,\, \mathcal{L}_m\right] &= (n-m)\mathcal{L}_{n+m} + \frac{D-2}{12}\,n(n^2-1)\delta_{n+m} \\ \left[\tilde{\mathcal{L}}_n,\, \tilde{\mathcal{L}}_m\right] &= (n-m)\tilde{\mathcal{L}}_{n+m} + \frac{D-2}{12}\,n(n^2-1)\delta_{n+m} \\ \left[\mathcal{L}_n,\, \tilde{\mathcal{L}}_m\right] &= 0\end{aligned} \tag{52}$$

these relations, apart from the term $\frac{D-2}{12}\,n(n^2-1)\delta_{n+m}$ on the left hand sides, are analogous to the relations (48) characterizing the generators of conformal transformations (Virasoro algebra).

Consider now the quantum version of the solution to the constraints in the light-come gauge, namely:

<u>open string</u>:

$$\alpha^-_n = \frac{1}{p^+\ell}\,(\mathcal{L}_n - a\,\delta_n) \;;\; \alpha^+_n = 0 \;(n \neq 0)$$

<u>closed string</u>:

$$\begin{aligned}\alpha^-_n &= \frac{2}{p^+\ell}\,(\mathcal{L}_n - a\,\delta_n) \;;\; \alpha^+_n = 0 \;(n \neq 0) \\ \tilde{\alpha}^-_n &= \frac{2}{p^+\ell}\,\left(\tilde{\mathcal{L}}_n - a\,\delta_n\right) \;;\; \tilde{\alpha}^+_n = 0 \;(n \neq 0)\end{aligned} \tag{53}$$

Recalling that in the open string case $\alpha_0^\mu = \ell p^\mu$ while in the closed string case $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \ell p^\mu/2$, express in both cases $M^2 = -p^\mu p_\mu$ as function of α_n^i, $\tilde{\alpha}_n^i$ and a and in terms of the operator N defined for the open string case in equation (40) and for the closed string case by:

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \left(\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \right) \tag{54}$$

What is the lowest value of M^2 in both cases? Show that in the case $a = 1$ these states are tachyons.

Derive in the open string case all the commutation relations between x^μ, p^μ and α_n^μ represented in the table in section 4.3 of the lecture notes.

Consider now the following quantum version of the expressions found in exercise 11 for the coefficients L_n and $\tilde{L}_n$ of the Fourier expansion of $T_{\pm\pm}$ in terms of the coefficients α_n and $\tilde{\alpha}_n$:

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_m^\mu \alpha_{n-m\,\mu} : \\ \tilde{L}_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m\,\mu} : \end{aligned} \tag{55}$$

Using equations (53) show that in the light-cone gauge $L_n = \tilde{L}_n = a\,\delta_n$.

Consider the open string case in which $\mathcal{L}_n = \tilde{\mathcal{L}}_n$. Show that the following relations hold:

$$\left[\alpha_n^-, \alpha_m^-\right] = (n-m)\frac{\alpha_{n+m}^-}{p^+\ell} + \frac{1}{(p^+\ell)^2}\left(\frac{D-2}{12}n(n^2-1) + 2na\right)\delta_{n+m} \tag{56}$$

16. Find values of the mass squared M^2 for the following states using the value $a = 1$:

<u>open string:</u>

$$|\psi_1\rangle = \alpha_{-1}^i|0\rangle$$
$$|\psi_2\rangle = \alpha_{-1}^i \alpha_{-1}^i \alpha_{-2}^j|0\rangle$$

closed string:

$$
\begin{aligned}
|\chi_1\rangle &= \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \\
|\chi_2\rangle &= \alpha^i_{-1}\alpha^i_{-1}\tilde{\alpha}^j_{-2}|0\rangle
\end{aligned}
\tag{57}
$$