13. Let us restrict ourselves to the quantized open string case in the light-cone gauge. Derive the commutation relations between x^i, p^i and α_n^i from equations (4.5) and (4.3) of the lecture notes. Let us define the vacuum state $|p^\mu, 0\rangle$ such that $\alpha_n^i|p^\mu, 0\rangle = 0$ for $n < 0$ and $\alpha_0^i|p^\mu, 0\rangle = p^i|p^\mu, 0\rangle$. A generic open string state is obtained by applying the creation operators α_n^i $(n < 0)$ a finite number of times to $|p^\mu, 0\rangle$. Consider the operator N defined as follows:

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i \tag{1}$$

Show that on the following state:

$$|p^\mu, N_{i,n}\rangle = \overbrace{\alpha_{-n}^i \ldots \alpha_{-n}^i}^{N_{i,n}} |p^\mu, 0\rangle \tag{2}$$

the operator N has the following value:

$$N\,|p^\mu, N_{i,n}\rangle = nN_{i,n}|p^\mu, N_{i,n}\rangle \tag{3}$$

14. A solution of the string equations of motion can be written in the form (28). For the open string we have the further constraint that $X_L^\mu(\tau) = X_R^\mu(\tau)$ and that $X_L^\mu(\sigma^+)$ and $X_R^\mu(\sigma^-)$ are respectively periodic in σ^+ and σ^- of period 2π a part for a constant shift while for the closed string X_L^μ and X_R^μ need only be periodic in $\sigma \in (0, \pi)$ at fixed τ. The theory is also invariant under conformal transformations which can be written in the following infinitesimal form:

$$\begin{aligned} \sigma^+ &\to \sigma^+ + \epsilon^+(\sigma^+) \\ \sigma^- &\to \sigma^- + \epsilon^-(\sigma^-) \end{aligned} \tag{4}$$

where $\epsilon^+(\sigma^+)$ and $\epsilon^-(\sigma^-)$ are periodic in σ^+ and σ^- of period 2π. For the open string we further require that $\epsilon^+ = \epsilon^-$. In this exercise we will compute the infinitesimal generators of conformal transformations and their algebraic properties. Being periodic of period 2π in their argument, the infinitesimal functions can be expanded in Fourier series:

$$\epsilon^\pm(\sigma^\pm) = i \sum_{n=-\infty}^{+\infty} \epsilon_n^\pm e^{in\sigma^\pm} \tag{5}$$

where the reality condition implies that $\epsilon^{\pm}_{-n} = -(\epsilon^{\pm}_n)^\star$.

A generic scalar function $Y(\sigma^\pm)$ will transform under (4) as follows:

$$\begin{aligned}
Y(\sigma^\pm) &\rightarrow Y(\sigma^\pm + \epsilon^\pm) \sim Y(\sigma^\pm) + \delta Y(\sigma^\pm) \\
\delta Y(\sigma^\pm) &= \epsilon^\pm \partial_\pm Y(\sigma^\pm) = i \sum_{n=-\infty}^{+\infty} \epsilon^\pm_n e^{in\sigma^\pm} \partial_\pm Y(\sigma^\pm)
\end{aligned} \tag{6}$$

where we have used eq. (5) to derive the last expression. Let us define the following differential operators $\mathbf{L}^{(\pm)}_n$ on functions of $\sigma^\pm$ respectively:

$$\begin{aligned}
\mathbf{L}^{(+)}_n \left(Y(\sigma^+)\right) &= ie^{in\sigma^+} \partial_+ Y(\sigma^+) \\
\mathbf{L}^{(-)}_n \left(Y(\sigma^-)\right) &= ie^{in\sigma^-} \partial_- Y(\sigma^-) \\
\mathbf{L}^{(+)}_n \left(Y(\sigma^-)\right) &= 0 \\
\mathbf{L}^{(-)}_n \left(Y(\sigma^+)\right) &= 0
\end{aligned} \tag{7}$$

the product of two of these operators is defined by their consecutive action on a same function Y: $\mathbf{L}_n \, \mathbf{L}_m \, (Y) \equiv \mathbf{L}_n \, (\mathbf{L}_m \, (Y))$. The operators $\mathbf{L}^{(\pm)}_n$ are the generators of the infinitesimal conformal transformations (6), indeed we can write:

$$\delta Y(\sigma^\pm) = \sum_{n=-\infty}^{+\infty} \epsilon^\pm_n \, \mathbf{L}^{(\pm)}_n \, (Y) \tag{8}$$

- Show that taking $Y(\sigma^\pm) = \sigma^\pm$ and considering the definitions (7) the infinitesimal conformal transformation (4) follows from (8).
- Show that on a generic function $Y(\sigma^\pm)$ the following relations hold:

$$\begin{aligned}
\left(\mathbf{L}^{(+)}_n \, \mathbf{L}^{(+)}_m - \mathbf{L}^{(+)}_m \, \mathbf{L}^{(+)}_n\right) \left(Y(\sigma^+)\right) &= (n-m)\mathbf{L}^{(+)}_{n+m} \left(Y(\sigma^+)\right) \\
\left(\mathbf{L}^{(-)}_n \, \mathbf{L}^{(-)}_m - \mathbf{L}^{(-)}_m \, \mathbf{L}^{(-)}_n\right) \left(Y(\sigma^-)\right) &= (n-m)\mathbf{L}^{(-)}_{n+m} \left(Y(\sigma^-)\right) \\
\left(\mathbf{L}^{(+)}_n \, \mathbf{L}^{(-)}_m - \mathbf{L}^{(-)}_m \, \mathbf{L}^{(+)}_n\right) \left(Y(\sigma^\pm)\right) &= 0
\end{aligned} \tag{9}$$

- Consider the following two constant infinitesimal transformations:

$$\begin{aligned}
\delta\tau &= c \; ; \; \delta\sigma = 0 \quad \text{time shift} \\
\delta\tau &= 0 \; ; \; \delta\sigma = c \quad \text{world sheet rotation}
\end{aligned} \tag{10}$$

Show that they are generated respectively by the following differential operators:

$$\begin{aligned}\mathcal{O}_t &= -i\,c\,(\mathbf{L}_0^{(+)} + \mathbf{L}_0^{(-)}) \\ \mathcal{O}_s &= -i\,c\,(\mathbf{L}_0^{(+)} - \mathbf{L}_0^{(-)}) \end{aligned} \tag{11}$$

namely that for the time shift $\delta\tau = \mathcal{O}_t(\tau) = c$ and $\delta\sigma = \mathcal{O}_s(\sigma) = 0$ while for the world sheet rotation $\delta\tau = \mathcal{O}_s(\tau) = 0$ and $\delta\sigma = \mathcal{O}_s(\sigma) = c$

15. Let us define the following operators on the closed string states:

$$\begin{aligned}\mathcal{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\alpha_m^i \alpha_{n-m}^i: \\ \tilde{\mathcal{L}}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i: \end{aligned} \tag{12}$$

where as usual α_n and $\tilde{\alpha}_n$ denote the right and left moving mode operators. Show that they fulfill the following commutation relations:

$$\begin{aligned}[\mathcal{L}_n,\, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + \frac{D-2}{12}\, n(n^2-1)\delta_{n+m} \\ \left[\tilde{\mathcal{L}}_n,\, \tilde{\mathcal{L}}_m\right] &= (n-m)\tilde{\mathcal{L}}_{n+m} + \frac{D-2}{12}\, n(n^2-1)\delta_{n+m} \\ \left[\mathcal{L}_n,\, \tilde{\mathcal{L}}_m\right] &= 0 \end{aligned} \tag{13}$$

these relations, apart from the term $\frac{D-2}{12}\, n(n^2-1)\delta_{n+m}$ on the left hand sides, are analogous to the relations (9) characterizing the generators of conformal transformations (Virasoro algebra).

Consider now the quantum version of the solution to the constraints in the light-come gauge, namely:

open string:

$$\alpha_n^- = \frac{1}{p^+\ell}\,(\mathcal{L}_n - a\,\delta_n)\ ;\ \ \alpha_n^+ = 0\ (n \neq 0)$$

closed string:

$$\begin{aligned}\alpha_n^- &= \frac{2}{p^+\ell}\,(\mathcal{L}_n - a\,\delta_n)\ ;\ \ \alpha_n^+ = 0\ (n \neq 0) \\ \tilde{\alpha}_n^- &= \frac{2}{p^+\ell}\left(\tilde{\mathcal{L}}_n - a\,\delta_n\right)\ ;\ \ \tilde{\alpha}_n^+ = 0\ (n \neq 0) \end{aligned} \tag{14}$$

Recalling that in the open string case $\alpha_0^\mu = \ell p^\mu$ while in the closed string case $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \ell p^\mu/2$, express in both cases $M^2 = -p^\mu p_\mu$ as function of α_n^i, $\tilde{\alpha}_n^i$ and a and in terms of the operator N defined for the open string case in equation (1) and for the closed string case by:

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \left(\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \right) \tag{15}$$

What is the lowest value of M^2 in both cases? Show that in the case $a = 1$ these states are tachyons.

Derive in the open string case all the commutation relations between x^μ, p^μ and α_n^μ represented in the table in section 4.3 of the lecture notes.

Consider now the following quantum version of the expressions found in exercise 11 for the coefficients L_n and $\tilde{L}_n$ of the Fourier expansion of $T_{\pm\pm}$ in terms of the coefficients α_n and $\tilde{\alpha}_n$:

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} :\alpha_m^\mu \alpha_{n-m\,\mu}: \\ \tilde{L}_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} :\tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m\,\mu}: \end{aligned} \tag{16}$$

Using equations (14) show that in the light-cone gauge $L_n = \tilde{L}_n = a\,\delta_n$.

Consider the open string case in which $\mathcal{L}_n = \tilde{\mathcal{L}}_n$. Show that the following relations hold:

$$\left[\alpha_n^-, \alpha_m^-\right] = (n-m)\frac{\alpha_{n+m}^-}{p^+\ell} + \frac{1}{(p^+\ell)^2}\left(\frac{D-2}{12} n(n^2-1) + 2na\right)\delta_{n+m} \tag{17}$$

16. Find values of the mass squared M^2 for the following states using the value $a = 1$:

<u>open string:</u>

$$|\psi_1\rangle = \alpha_{-1}^i |0\rangle$$
$$|\psi_2\rangle = \alpha_{-1}^i \alpha_{-1}^i \alpha_{-2}^j |0\rangle$$

closed string:

$$
\begin{aligned}
|\chi_1\rangle &= \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \\
|\chi_2\rangle &= \alpha^i_{-1}\alpha^i_{-1}\tilde{\alpha}^j_{-2}|0\rangle
\end{aligned} \tag{18}
$$

17. Consider the transverse components of the open string operator $X^{tr}(\tau,\sigma) = X^i(\tau,\sigma)$, $(i = 1,\dots,D-2)$. We wish to compute the propagator associated with the transverse modes of an open string. Let us extend for convenience the definition of normal ordering to the operators p^i, x^i in the following way:

$$
\begin{aligned}
:\alpha^i_n\alpha^j_{-n}: &= :\alpha^j_{-n}\alpha^i_n: = \alpha^j_{-n}\alpha^i_n\ ;\ \ (n>0) \\
:p^i x^j: &= :x^j p^i: = x^j p^i
\end{aligned} \tag{19}
$$

Show that:

$$
\begin{aligned}
X^i(\tau,\sigma)X^j(\tau',\sigma') &= :X^i(\tau,\sigma)X^j(\tau',\sigma'):+\delta^{ij}\,G(\tau,\,\tau',\,\sigma,\,\sigma') \\
G(\tau,\,\tau',\,\sigma,\,\sigma') &= -\frac{1}{4}\log\left[(e^{i\sigma'^+}-e^{i\sigma^+})(e^{i\sigma'^-}-e^{i\sigma^-})\right]+ \\
&\quad \mathrm{R}\,(\tau\to\tau',\sigma\to\sigma')
\end{aligned} \tag{20}
$$

where "$\mathrm{R}\,(\tau\to\tau',\sigma\to\sigma')$" denotes terms which are not divergent in the limit $\tau\to\tau',\sigma\to\sigma'$. Find the expressions of G and therefore of $\mathrm{R}\,(\tau\to\tau',\sigma\to\sigma')$.

hint: *write the open string solution $X^i(\tau,\sigma)$ in the form (taking the scale $\ell = 1$):*

$$
\begin{aligned}
X^i(\tau,\sigma) &= x^i(\tau)+A^i(\tau,\sigma)+A^{i\dagger}(\tau,\sigma) \\
x^i(\tau) &= x^i + p^i\tau \\
A^i(\tau,\sigma) &= i\sum_{n=1}^{\infty}\frac{\alpha^i_n}{n}\,e^{-in\tau}\,\cos(n\sigma) \\
A^{i\dagger}(\tau,\sigma) &= -i\sum_{n=1}^{\infty}\frac{\alpha^i_{-n}}{n}\,e^{in\tau}\,\cos(n\sigma)
\end{aligned} \tag{21}
$$

show that the only non normal ordered terms in the product $X^i(\tau,\sigma)X^j(\tau',\sigma')$ are $p^i x^j\tau$ and $A^i(\tau,\sigma)\,A^{i\dagger}(\tau',\sigma')$. These terms will be rewritten in terms *of their normal products and of the following commutators:* $[p^i, x^j]\,\tau$ *and* $\left[A^i(\tau,\sigma),\, A^{j\dagger}(\tau',\sigma')\right]$. *Using the formula:*

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \;=\; -\log\left(1-x\right) \qquad (22)$$

from the expression of these commutators deduce $G(\tau,\, \tau',\, \sigma,\, \sigma')$.

Show that for $\tau \neq \tau'$ and $\sigma \neq \sigma'$, $G(\tau,\, \tau',\, \sigma,\, \sigma')$ fulfills the equation of motion:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, G(\tau,\, \tau',\, \sigma,\, \sigma') \;=\; 0 \qquad (23)$$

where the partial derivation is ment *only* with respect to τ, σ and not with respect to τ', σ'. From equations (20) show that the function G is the *propagator* of each transverse mode from the point (τ',σ') to the point (τ,σ), namely show that for $\tau > \tau'$:

$$\langle 0|X^i(\tau,\sigma)X^j(\tau',\sigma')|0\rangle \;=\; \delta^{ij}\, G(\tau,\, \sigma,\, \tau',\, \sigma')\;\;(\tau > \tau') \qquad (24)$$

In the following we shall always consider $\tau > \tau'$.

We wish to show that $G(\tau,\, \tau',\, \sigma,\, \sigma')$ in the whole τ, σ plane fulfills the following equation:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, G(\tau,\, \tau',\, \sigma,\, \sigma') \;=\; i\pi\, \delta(\tau-\tau')\delta(\sigma-\sigma') \qquad (25)$$

Only the terms in G which diverge for $\tau \to \tau'$ and $\sigma \to \sigma'$ (see equation (20)) contribute to the delta functions on the right hand side while the regular terms in $\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$ are fixed by other boundary conditions which we shall not consider here. Let us introduce the Green function $\tilde{G}(\tau-\tau',\, \sigma-\sigma')$ of the open string field equation:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, \tilde{G}(\tau-\tau',\, \sigma-\sigma') \;=\; \delta(\tau-\tau')\delta(\sigma-\sigma') \qquad (26)$$

It is useful to express a solution of the above equation in terms of Fourier transforms with respect to the variables $\tau - \tau' \in (-\infty,\, \infty)$ and $\sigma - \sigma' \in (-\pi,\, \pi)$. Let us define the following new variables: $\xi^0 = (\tau-\tau')\Delta$, $\xi^1 = (\sigma-\sigma')\Delta$ where Δ is a scale which we shall send to infinity in order to work with coordinates ξ^α which run from $-\infty$ to $+\infty$ (indeed ξ^1 will take values in the interval $(-\pi\Delta,\, \pi\Delta)$ which becomes

$(-\infty, \infty)$ in the limit $\Delta \to \infty$). The momenta associated with ξ^α are denoted by k^α. In particular k^1 is quantized as n/Δ (n integer) and in the limit $\Delta \to \infty$ becomes a continuous variable. Therefore the following approximation holds:

$$\sum_{n=0}^{\infty} f(\sigma - \sigma')\, e^{in(\sigma-\sigma')} \quad \sim \quad \Delta \int_0^\infty dk^1\, f(\xi^1/\Delta)\, e^{ik^1\xi^1} \tag{27}$$

Show that the following function:

$$\begin{aligned} \tilde{G}(\tau - \tau', \sigma - \sigma') &= -\int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon} \\ \xi^\alpha &= \Delta(\sigma^\alpha - \sigma'^\alpha) \end{aligned} \tag{28}$$

fulfills equ. (26) by using the following useful relations:

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{ixy} &= 2\pi\, \delta(y) \\ \delta(ay) &= \frac{1}{a}\delta(y) \end{aligned} \tag{29}$$

(ϵ is an infinitesimal parameter used for regularizing the integral).

Let us show that the singular parts of G and of $\tilde{G}$ as given by equ. (28) are proportional. In this way we would show that G is proportional to the Green function of the open string equation of motion.

We start by computing $\tilde{G}$ from equ. (28) as a function of the world sheet coordinates. Show that:

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon} \quad = \quad -\frac{i}{2(|k^1| - i\epsilon)}\, e^{-i|k^1|\xi^0 + ik^1\xi^1} \tag{30}$$

hint: *compute the integral in the complex k^0 plane and write the denominator as the product of two simple poles:*

$$(k^0)^2 - (k^1)^2 + i\epsilon \quad = \quad \left[k^0 - (|k^1| - i\epsilon)\right]\left[k^0 - (-|k^1| + i\epsilon)\right] \tag{31}$$

since $\xi^0 > 0$ we should close the contour of integration in the lower plane so to include the pole $|k^1| - i\epsilon$ and then compute the integral as the residue in that pole using the formula:

$$\oint_{C_{z_0}} dz \frac{f(z)}{z - z_0} \quad = \quad -2\pi i f(z_0) \tag{32}$$

where $f(z)$ is a regular function in z_0 and C_{z_0} is a contour around z_0 oriented clockwise.

We are left with the integral over k^1 which is divergent in $k^1 = 0$ in the limit $\epsilon \to 0$. It is useful to express this integral in terms of its principal part by using the relation:

$$\frac{1}{z \pm i\epsilon} = \mathrm{PP}\left(\frac{1}{z}\right) \mp i\delta(z) \tag{33}$$

Show that the second term on the right hand side contributes to $\tilde{G}$ with a constant term. Since we are interested only in the singular part of $\tilde{G}$ we shall ignore this term and simply substitute the integral in k^1 with its principal part which can be expressed as follows:

$$\mathrm{PP}\int_{-\infty}^{\infty} dk^1\, f(k^1) = \lim_{\Delta\to\infty}\left(\int_{1/\Delta}^{\infty} dk^1\, f(k^1) + \int_{-\infty}^{-1/\Delta} dk^1\, f(k^1)\right)$$

Show that:

$$\begin{aligned}\tilde{G}(\tau-\tau',\, \sigma-\sigma') &= \mathrm{PP}\int_{-\infty}^{\infty} \frac{dk^1}{2\pi}\,\frac{i}{2|k^1|}\, e^{-i|k^1|\xi^0 + ik^1\xi^1} = \\ & \frac{i}{4\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}\, e^{-in(\sigma^- - \sigma'^-)} + \sum_{n=1}^{\infty}\frac{1}{n}\, e^{-in(\sigma^+ - \sigma'^+)}\right)\end{aligned} \tag{34}$$

hint: *use the property that $k^1 = n/\Delta$ and therefore for large Δ*

$$\int_{1/\Delta}^{\infty} dk^1 \sim (1/\Delta)\sum_{n=1}^{\infty}$$

From equ. (34) show that the functions $\tilde{G}$ and G are proportional a part from terms which are not divergent in the limit $\tau \to \tau'$ and $\sigma \to \sigma'$ and from this deduce that equation (25) holds.

18. Show that:

$$\begin{aligned} :e^{ik_1^i X^i(\tau,0)} :: e^{ik_2^i X^i(0,0)}: &= :e^{ik_1^i X^i(\tau,0)}\, e^{ik_2^i X^i(0,0)}:\, e^{-(k_1\cdot k_2)\, G(\tau,0,0,0)} = \\ & (1-x)^{k_1\cdot k_2}\, x^{-k_1\cdot k_2} \\ x &= e^{-i\tau} \end{aligned} \tag{35}$$

by using the expression for the propagator G found in the previous exercise and the basic formulas in section 6 of the lecture notes