19. Anticommuting c-numbers

Consider anticommuting c-numbers θ_i, satisfying

$$\theta_i \theta_j = -\theta_j \theta_i \tag{1}$$

The numbers θ_i are taken to be real, i.e. $\theta_i^\dagger = \theta_i$.

a) Define a function $F(\theta_1, \theta_2, \theta_3)$, where the θ_i are 3 anticommuting variables, called *Grassmann* variables. Show that there are 8 terms (*monomials*) in the polynomial decomposition of the function in terms of the θ variables. How many independent terms do you expect in the decomposition of a function of n Grassmann variables?

We first look at differentiation. The *left* derivative of a function is obtained by differentiating its monomials and resumming the result. To calculate the left derivative with respect to θ_i we must, in every monomial, permute θ_i to the left and then drop it. Let $\epsilon[i]$ be the sign of the permutation needed to bring θ_i to the left, and $\epsilon[i] = 0$ when θ_i does not occur in the monomial. The left derivative of a monomial can then be written in the following way,

$$\frac{\vec{\partial}}{\partial \theta_{i_k}} \left(\theta_{i_1} \cdots \theta_{i_{k-1}} \theta_{i_k} \theta_{i_{k+1}} \cdots \theta_{i_m}\right) = \epsilon[i_k] \left(\theta_{i_1} \cdots \theta_{i_{k-1}} \theta_{i_{k+1}} \cdots \theta_{i_m}\right). \tag{2}$$

Analogously, one can define a *right* derivative.

b) Define a function $F(\theta_1, \theta_2)$, calculate

$$\frac{\vec{\partial}}{\partial \theta_1} F(\theta_1, \theta_2), \text{ and}$$
$$\frac{\vec{\partial}}{\partial \theta_2} F(\theta_1, \theta_2).$$

The definition of the left derivative corresponds to the variational statement

$$\delta_\theta F(\theta) = \delta\theta \, \frac{\vec{\partial}}{\partial \theta} F(\theta), \tag{3}$$

where $\delta\theta$ is a Grassmann parameter. Note the order of the terms in the above expression.

c) For the right derivative write down the corresponding variational statement.

d) Consider a $\psi(\theta_i)$ which is a function of the θ_i $(i = 1, \dots, n)$ variables. Work out

$$\frac{\vec{\partial}}{\partial \theta_j} F(\psi(\theta_i)); \quad \frac{\overleftarrow{\partial}}{\partial \theta_j} \left(F(\psi(\theta_i))\right). \tag{4}$$

Again, the order in which the terms appear is important. This is an immediate consequence of working with anticommuting variables. Now look at Leibniz' rule:

e) Work out

$$\frac{\vec{\partial}}{\partial \theta} \left(F(\theta)\ G(\theta)\right). \tag{5}$$

f) Calculate

$$\frac{\vec{\partial}}{\partial \theta} \exp(\theta) \tag{6}$$

We now construct the analogue of the indefinite one-dimensional integral,

$$\int_{-\infty}^{\infty} dx\ f(x), \tag{7}$$

for the case of anticommuting c-numbers, which we denote by

$$\int d\theta F(\theta). \tag{8}$$

We want it to obey the following property of the integral over commuting variables,

$$\int_{-\infty}^{\infty} dx\ f(x) = \int_{-\infty}^{\infty} dx\ f(x+a) \tag{9}$$

with a finite.

g) Consider a function $F(\theta)$ of one Grassmann variable θ. Show that requiring the property (9) to hold leads to the requirement

$$\int d\theta\ [\text{any element not depending on } \theta] = 0. \tag{10}$$

Compare with differentiating a function of commuting variables.

We are then left with an integral over θ, which we normalize to unity,

$$\int d\theta\ \theta \equiv 1. \tag{11}$$

20. Bosons and fermions - change of variables

In field theory, we can often apply the methods we know from working with bosonic variables to fermionic variables as well. However, sometimes there are slight differences, usually minus signs, that one has to be very careful about. In this exercise we will show one of these differences; in the next exercise we will find another.

It is well-known from ordinary integral calculus that a linear change of variables $\boldsymbol{y} = A\boldsymbol{x}$ leads to a change in the measure:

$$d\boldsymbol{y} = (\det A)d\boldsymbol{x} \tag{12}$$

For anticommuting variables, this turns out not to be the case. Consider the integral

$$\int d\psi_1 d\psi_2 \, \psi_1\psi_2, \tag{13}$$

where the ψ_i are anticommuting variables, i. e. $\psi_i\psi_j = -\psi_j\psi_i$. Now apply a linear change of variables: $\boldsymbol{\chi} = M\boldsymbol{\psi}$.

h) Express $\chi_1\chi_2$ in terms of ψ_1 and ψ_2.

i) Using the fact that $\int d\psi_1 d\psi_2 \, \psi_1\psi_2 = \int d\chi_1 d\chi_2 \, \chi_1\chi_2$, show that in this case the measure must transform as $d\chi_1 d\chi_2 = (\det M)^{-1} d\psi_1 d\psi_2$. Note the power -1, which is different in the bosonic case.

j) Show that the same result holds when we start with $\psi_1, \dots, \psi_n$.

This result can even be generalized to the case of nonlinear changes of variables, where the matrix M is replaced by the Jacobian matrix J.

21. Feynman rules: the fermion loop

In this exercise, we will study a field theory given by the Lagrangian

$$S = \int dt \, \left(\phi D_b \phi + \bar{\psi} D_f \psi + \lambda\phi\bar{\psi}\psi\right), \tag{14}$$

where $\phi(t)$ is a commuting (bosonic) field, and $\psi(t)$ is anticommuting (fermionic). Note, that commuting fields *commute* with anticommuting fields, i. e. $[\psi(t), \phi(t')] = 0$. Here, D_b and D_f are differential operators.

By the same method as we used in the previous exercise, show that the path integral with source terms can be written as:

$$Z[J, \bar{\eta}, \eta] = \int D\phi D\bar{\psi} D\psi \, \exp\left(\tfrac{i}{\hbar}(S + J\phi + \bar{\eta}\psi + \bar{\psi}\eta)\right) \tag{15}$$

(where we used the shorthand $J\phi \equiv \int dt\, J(t)\phi(t)$, etc.), is equal to

$$\exp\left[\frac{i}{\hbar}\int dt\,\left(i\lambda\frac{\partial}{\partial J(t)}\frac{\partial}{\partial\bar{\eta}(t)}\frac{\partial}{\partial\eta(t)}\right)\right] Z_0[J,\bar{\eta},\eta], \tag{16}$$

where

$$\begin{aligned} Z_0[J,\bar{\eta},\eta] &= \exp\left[-\frac{i}{2}\int dtdt'\,(J(t)\Delta_b(t-t')J(t'))\right]\times \\ &\quad\times\exp\left[-i\int dtdt'\,(\bar{\eta}(t)\Delta_f(t-t')\eta(t'))\right], \end{aligned} \tag{17}$$

and $\Delta_{b,f}$ are the Green's functions that correspond to $D_{b,f}$.

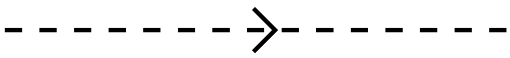

Figure 1: The fermion propagator

Again, we can expand the exponantial acting on Z_0, and obtain solutions that are power series in λ. Just like in the previous exercise, we can draw Feynman diagrams to 'summarize' the solution.

k) Draw the fermion propagator as in Fig. 1. What does the arrow mean? Draw all diagrams that correspond to λ^2-terms, and give their combinatorial factors.

In trying to calculate the expression (16) (or in fact in deriving it), we encounter a problem: what is the exact definition of $\partial/\partial\eta(t)$?

If we did not define the derivative as in the previous exercise to be the left derivative we would obtain contradictory results $\alpha\beta = -\beta\alpha$ but $\partial/\partial\alpha(\alpha\beta) = \beta = \partial/\partial\alpha(\beta\alpha)$.

So studying this problem we need to apply the same prodecure as in the previous exercise: if we want to calculate $\partial/\partial\alpha$ of some functions, we first anticommute α to the left, and then use the ordinary rules of differentiation.

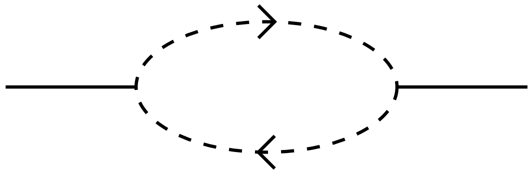

Figure 2: A Feynman diagram containing a fermion loop.

Knowing this, we will study the diagram that is drawn in Fig. 2.

l) Give the explicit expression that corresponds to this diagram. Hint: to get the minus signs right, write the exponential in the fermionic fields in Z_0 as a power series.

m) Compare this to the similar diagram we would obtain in a theory with a $\lambda\phi^3$-interaction instead of a $\lambda\phi\bar{\psi}\psi$-interaction. There are *two* differences, but only one comes from the fermionic nature of the variables.

n) Would it make any difference if we would have used a right derivative instead of the left one? Note, that in this case the exponential in Eq. (16) acts from the right.

o) What other diagrams that we obtained in (a) will obtain a relative minus-sign when compared to ϕ^3-theory? Convince yourself that the minus sign is related to the difference in the relation between the bosonic and fermionic path integrals with the corresponding determinants.