22. Let us consider the change of variables from τ, σ to z, $\overline{z}$. After a Wick rotation we write $\tau = i\sigma_2$ where σ_2 is the real Euclidean time and we rename σ as σ_1. If we define $w = \sigma^+ = i\sigma_2 + \sigma_1$ so that $\overline{w} = -\sigma^- = -i\sigma_2 + \sigma_1$ then z, $\overline{z}$ are defined as follows:

$$\begin{array}{lll} \underline{\text{open string:}} & z = e^{iw} \; ; & \overline{z} = e^{-i\overline{w}} \\ \underline{\text{closed string:}} & z = e^{2iw} \; ; & \overline{z} = e^{-2i\overline{w}} \end{array} \qquad (1)$$

- For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by z for the open and closed string cases and draw the equal–time curves. Which points correspond to $\sigma_2 \to \pm\infty$?
- Write as functions of the complex variables the open and closed string solutions $X^\mu(z, \overline{z})$ together with their holomorphic and anti–holomorphic derivatives $\partial_z X^\mu(z, \overline{z})$, $\partial_{\overline{z}} X^\mu(z, \overline{z})$.
- Rewrite the Green function $G(\tau, \tau', \sigma, \sigma')$ computed in exercise 17 as a function $G(z, \overline{z}, z', \overline{z}')$ in the complex variables. Compute it for the closed string as well.
- Consider the variable $\hat{z} = \log(z)$. For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by $\hat{z}$ for the open and closed string cases and draw the equal–time curves.

23. Let us consider conformal transformations in the complex coordinate notation. In these coordinates conformal transformations have the form: $z \to z'(z)$ and $\overline{z} \to \overline{z}'(\overline{z})$. We shall denote in the sequel by $\Phi_{h,\bar{h}}(z, \overline{z})$ a tensor with h holomorphic and $\bar{h}$ anti–holomorphic lower indices. Under a conformal transformation $\Phi_{h,\bar{h}}$ transforms as follows:

$$\Phi_{h,\bar{h}}(z, \overline{z}) \quad \to \quad \Phi'_{h,\bar{h}}(z, \overline{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{\bar{h}} \Phi_{h,\bar{h}}(z'(z), \overline{z}'(\overline{z})) \qquad (2)$$

- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to e^{i\theta} z$, $\overline{z} \to e^{-i\theta}\, \overline{z}$ where θ is a constant angle.
- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to z' = \log(z)$, $\overline{z} \to \overline{z}' = \log(\overline{z})$.

Consider an infinitesimal conformal transformation $z \to z' = z + \epsilon(z)$ and $\overline{z} \to \overline{z}' = \overline{z} + \overline{\epsilon}(\overline{z})$. Show that:

$$\begin{aligned} \Phi'_{h,\bar{h}} &= \Phi_{h,\bar{h}} + \delta\Phi_{h,\bar{h}} \\ \delta\Phi_{h,\bar{h}} &= \left(h\,(\partial\epsilon) + \bar{h}\,(\overline{\partial}\overline{\epsilon}) + \epsilon\,\partial + \overline{\epsilon}\,\overline{\partial}\right) \Phi_{h,\bar{h}} \end{aligned} \qquad (3)$$

where we have used the notation: $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$.

Consider now the quantized closed string and let $\Phi_{h,\bar{h}}$ be an operator on the Hilbert space of states which transforms under conformal transformations as in eq (2) (*primary operator*) . We wish to define generators of infinitesimal conformal transformations T_ϵ, $\overline{T}_{\overline{\epsilon}}$ on the Hilbert space such that:

$$\delta\Phi_{h,\bar{h}} \;=\; [T_\epsilon,\, \Phi_{h,\bar{h}}] + [\overline{T}_{\overline{\epsilon}},\, \Phi_{h,\bar{h}}] \tag{4}$$

Let us consider the Laurent expansion of ϵ, $\overline{\epsilon}$:

$$\epsilon = \sum_{n=-\infty}^{\infty} \epsilon_n\, z^{n+1} \qquad \overline{\epsilon} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \overline{z}^{n+1}$$

and define T_ϵ, $\overline{T}_{\overline{\epsilon}}$ in terms of some operators ℓ_n and $\bar{\ell}_n$ as follows:

$$T_\epsilon \;=\; \sum_{n=-\infty}^{\infty} \epsilon_n\, \ell_n \quad \overline{T}_{\overline{\epsilon}} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \bar{\ell}_n \tag{5}$$

Show that if the following commutation relations hold:

$$\begin{aligned} \left[\ell_n,\, \Phi_{h,\bar{h}}\right] &= z^n(z\partial + h(n+1))\Phi_{h,\bar{h}} \\ \left[\bar{\ell}_n,\, \Phi_{h,\bar{h}}\right] &= \overline{z}^n(\overline{z}\overline{\partial} + \bar{h}(n+1))\Phi_{h,\bar{h}} \end{aligned} \tag{6}$$

formula (4) yields the transformation rule (3). Compare formulas (6) and (5) with formula (47) of exercise 14. Observe the differences with respect to exercise 14: there conformal transformations were considered for simplicity only on scalar functions Y (which have $h = \bar{h} = 0$); the infinitesimal generator had the same form as in (5), but its action on the function Y is a differential operation, while in the present exercise its action on operators is expressed in terms of commutators (6); the light cone indices $\pm$ are substituted by holomorphic/anti–holomorphic indices in the present exercise.

Operators on free closed string states are expressed in terms of α_n^i and $\tilde{\alpha}_n^i$. An asymptotic string state emitted from a point $z, \overline{z}$ of the world sheet is described by a local *primary* operator $V(z, \overline{z})$ (i.e. transforming as in equ.(2) with definite values of h, $\bar{h}$) which is called *vertex operator*.

Consider the vertex operator $V(z,\overline{z}) =: e^{ik^i X^i(z,\overline{z})}:$ (from now on, for the vertex operators, we shall consider only space–like momenta $k^\mu \equiv k^i$) describing the emission of a tachyon of momentum k^i (verify indeed that $[p^i, V] = k^i\, V$). If for ℓ_n and $\bar{\ell}_n$ we take $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ defined in exercise 15, show that the commutator of these operators with V have the expression on the right hand side of eqs. (6) for suitable values of h, $\bar{h}$. Find these values. Repeat this exercise for the vertex operator $V(z,\overline{z}) =: \partial X^j\, \overline{\partial} X^i\, e^{ik\cdot X(z,\overline{z})}:$.

In general infinitesimal conformal transformations on the closed string Hilbert space are generated by T_ϵ, $\overline{T}_{\overline{\epsilon}}$ which are expressed in terms of $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$.

There is a remark to be done about conformal transformations and the light–cone gauge. In the light–cone gauge the reparametrization invariance of the string action is totally fixed and with it also the conformal transformations which are generated by T_{++} and T_{--}. Indeed on the Hilbert space the Fourier modes of T_{++} and T_{--}, which are L_n and $\tilde{L}_n$ and are the generators of the conformal transformations are fixed to the value $\delta_{n,0}$ (see exercise 15). The generators $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ have the same expression as L_n and $\tilde{L}_n$ but involve only oscillators along transverse directions. They close a Virasoro algebra as well but with a different central charge given by $c = D - 2$, and thus are generators of conformal transformations though we do not expect the action to be invariant with respect to it (since all reparametrization invariance have been fixed).

In the framework of covariant quantization the constraint $T_{\alpha\beta} = 0$ is imposed on the physical states and translated into the conditions: $L_n|phys\rangle = \tilde{L}_n|phys\rangle = 0$ $(n > 0)$; $(L_0 - 1)|phys\rangle = (\tilde{L}_0 - 1)|phys\rangle = 0$. To summarize the $\{L_n,\, \tilde{L}_n\}$ and $\{\mathcal{L}_n,\, \tilde{\mathcal{L}}_n\}$ conformal algebras are different and with respect to them physical states will have different conformal weights $\{h,\, \bar{h}\}$. In particular the conformal weights with respect to the first are bound by the constraints written above to be $(1, 1)$, while the conformal weights corresponding to the action of the second algebra are not constrained.

24. There is a one to one correspondence between vertex operators $V(z, \overline{z})$ and asymptotic in–coming or our–going states of a string (free states). The in–coming string state $|V,\, in\rangle$ associated with the operator $V(z, \overline{z})$ is defined through the asymptotic limit $\sigma_2 \to \infty$ of $V(z, \overline{z})|0\rangle$ (recall that $|0\rangle$ is the vacuum state with vanishing CM momentum), similarly the corresponding out–going free state $\langle V,\, out|$ is obtained by performing on the state $\langle 0|V(z, \overline{z})$ the opposite limit $\sigma_2 \to -\infty$. Use the complex notation and find the in–coming and out–going states corresponding to the following vertex operators (for simplicity in what follows the momentum k^μ is always space-like):

$$\begin{array}{ll} \underline{\text{open string:}} & :e^{ik^i X^i(z,\overline{z})}: \\ & :\partial X^j\, e^{ik\cdot X(z,\overline{z})}: \\ & \\ \underline{\text{closed string:}} & :e^{ik^i X^i(z,\overline{z})}: \\ & :\partial X^j\, \overline{\partial} X^i\, e^{ik\cdot X(z,\overline{z})}: \end{array} \tag{7}$$

Compute the mass squared M^2 of the in–coming asymptotic states computed above either through direct evaluation of M^2 on $|V,\, in\rangle$ or by performing the limit $\sigma_2 \to \infty$ on $M^2(V(z, \overline{z})|0\rangle)$.

hint: *express* $M^2(V(z, \overline{z})|0\rangle)$ *as* $[M^2,\, V(z, \overline{z})]\,|0\rangle + V(z, \overline{z})\, M^2|0\rangle$