25. Representations of the Clifford algebra The Clifford algebra of Dirac matrices:

$$\{\gamma^\mu \gamma^\nu\} = -2\eta^{\mu\nu} \tag{1}$$

cannot be represented in any dimensions. Show, that for representing the γ^μ matrices in $2n$ spacetime dimensions we need at least $2^n \times 2^n$ matrices. (Hint: Construct n linear operators out of the γ matrices, eg.

$$a_i := \frac{1}{\sqrt{2}}(\gamma_{2i-2} + i\gamma_{2i-1}) \quad i = 1..n$$

use a representation [1]for which

$$\gamma^\dagger = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i \quad i = 1..2n-1.$$

Show that

$$\{a_n, a_m^\dagger\} = \delta_{m,n},$$

define a vacuum $|0\rangle$ and determine the number of different states which the creation operators can create from the vacuum. If we want to impose further conditions (like Weyl or Majorana spinor) we will find further constraints. Check that for two dimensions the following definition

$$\gamma^0 = \sigma^2 \quad \gamma^1 = i\sigma^1$$

(where σ^i means the i-th Pauli matrix) satisfies the algebra (1). Now with an inductive procedure we can construct a representation of 2^n dimensions if the spacetime has $2n$ dimensions (that is we will have 2n $2^n \times 2^n$ matrices). Suppose that we have the algebra for $d = 2n-2$.($\hat{\gamma}^\mu$'s are given for $\mu = 0...2n-3$ and (1) is satisfied). Then we define two more matrices according to the following:

$$\gamma^{2n-2} = i\mathbf{1} \times \sigma^1 \quad \gamma^{2n-1} = i\mathbf{1} \times \sigma^2 \tag{2}$$

and extend the first $2n-2$ as

$$\gamma^\mu = \hat{\gamma}^\mu \times (-\sigma_3) \quad \mu = 1..2n-1$$

$\mathbf{1}$ is the unit matrix in 2^{n-1} dimensions. ($(A \times B)_{ijkl} = A_{ik}B_{jl}$ is the formula for the indices of the direct product matrix.) That is these two matrices are blockdiagonal ones, each entry of the appropriate Pauli matrix is the entry times the $2^{n-1} \times 2^{n-1}$ $\hat{\gamma}^\mu$ or unit matrix. Check that (1) is then satisfied in 2n dimensions. What about the case of odd

[1]One needs to prove that this representation always exists.

dimensions ? Show that in $2n+1$ dimensions $2^n \times 2^n$ matrices still can represent the algebra. (Hint: Extend the case of $2n$ with $\gamma^{2n} = \gamma$ defined as (5) further in the exercise.

Since the right hand side of (1) is real the relation for the anticommutator of the complex conjugate γ's are the same:

$$\{\gamma^{\mu *}\gamma^{\nu *}\} = -2\eta^{\mu\nu}$$

The complex conjugate representation is equivalent, that is

$$\gamma^{\mu *} = B\gamma^{\mu}B^{-1} \quad \mu = 1...2n-1 \tag{3}$$

where both B_1 and B_2 are unitary. An explicit construction for such a matrix is the following:

$$B_1 = \gamma^3\gamma^5...\gamma^{2n-1} \quad B_2 = \gamma^0\gamma^1\gamma^2\gamma^4...\gamma^{2n-2} \tag{4}$$

Show that

$$B_1\gamma^{\mu}B_1^{-1} = (-1)^n\gamma^{\mu *} \quad B_2\gamma^{\mu}B_2^{-1} = (-1)^{n-1}\gamma^{\mu *}$$

that is one of them always satisfies (3).

Define now

$$\gamma = c\gamma^0\gamma^1...\gamma^{2n-1} \tag{5}$$

Determine the value of c by the requirement that $(\gamma)^2 = \mathbf{1}$. Can we generalize the relations (4) to odd $(2n+1)$ dimensions ?

The spinors on which the γ^μ matrices act are representations of the Lorentz group with the generator

$$\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

where the bracket is the commutator. With respect to the full Lorentz group these are not always irreducible representations, the familiar projections,

$$P_L = \frac{1-\gamma}{2} \quad P_R = \frac{1+\gamma}{2}$$

project out irreducible subspaces. In physical language we can find chiral (left and right handed) fermions, which are not mixed under the action of the Lorentz group. They are then called *Weyl spinors.*

When can one find real spinor representations ? In other words one wants to impose the following condition

$$\psi = B\psi^* \tag{6}$$

Show that ψ and $B\psi^*$ transforms according to the same representation of the Lorentz group. Show, that from the above condition requirement $B^*B = 1$ follows. Verify the following formulae for the explicit B_1 and B_2:

$$B_1 B_1^* = (-1)^{\frac{n(n-1)}{2}} \quad B_2 B_2^* = (-1)^{\frac{(n-1)(n-2)}{2}}$$

The spinors for which (6) holds are called *Majorana spinors*.

Verify that

$$B_i(\frac{1 \pm \gamma}{2}\psi)^* = \frac{1 \pm (-1)^{(n+1)}}{2} B_i \psi^* \quad i = 1, 2$$

that is the in certain spacetime dimensions

$$(\psi_L)^* = \psi_L \quad (\psi_R)^* = \psi_R$$

that is in certain spacetime dimensions $\psi_L^* = \psi_L$, $\psi_R^* = \psi_R^*$. In which spacetime dimensions are there Weyl, Majorana, Weyl-Majorana spinors ? (Hint: Use the given explicit representation of B's and γ^μ's.)

26. Check explicitly the calculations leading to (11.20) and (11.21) of the lecture notes.