27. **Supersymmetric particle.** Consider the action of a massless *super particle* propagating in D–dimensional Minkowski space, which is obtained from the particle world line action by adding to the D bosonic fields $x^\mu(\tau)$, the contribution from D Majorana fermions $\psi^\mu(\tau)$ ($\psi^{\mu\star} = \psi^\mu$) which describe the corresponding *super partner*:

$$S = -\frac{1}{2}\int d\tau\, e\, \left((\dot{x}^\mu)^2 - i\psi^\mu \dot{\psi}_\mu\right) \tag{1}$$

Compute the field equations for the fields x^μ, ψ^μ, e and show that the field equation for e amounts to a constraint on the other fields. Consider the following super symmetry transformation:

$$\delta x^\mu = i\epsilon\psi^\mu \;\; ; \quad \delta\psi^\mu = \epsilon\dot{x}^\mu \tag{2}$$

where ϵ is an infinitesimal real Grassmann parameter (recall that on Grassmann variables the complex conjugation is defined to have the property that $(\epsilon_1\epsilon_2)^* = \epsilon_2^*\epsilon_1^*$ just like Hermitian conjugation for matrices. Therefore you can show that the product of two real Grassmann variables acts as an imaginary c–number). The fields x^μ and ψ^μ are called super partners because they transform into each other by super symmetry. If we denote by δ_1 and δ_2 two infinitesimal super symmetry transformations parametrized by ϵ_1 and ϵ_2 respectively, using (2), show that:

$$\delta_1(\delta_2 x^\mu) - \delta_2(\delta_1 x^\mu) = \delta\tau\,\frac{dx^\mu}{d\tau} \tag{3}$$

and find the expression of $\delta\tau$. From the above result we see that the commutator of two super symmetry transformations amounts to a space–time coordinate translation. If the action were invariant under *local* super symmetry then (3) would guarantee its invariance under local reparametrization as well. Show that under the transformation (2) if ϵ depends on τ we can write:

$$\delta S = -2\int d\tau\,\left(i\dot{\epsilon}J - \frac{ie}{2}\frac{d}{d\tau}(\epsilon\dot{x}\psi) + \text{field equations}\right) \tag{4}$$

and find the expression for J. Is the action S invariant under global super symmetry transformations (i.e. $\dot{\epsilon} = 0$)? Under which constraint is the action invariant under local super symmetry transformations?

Since our action is invariant under local reparametrization, in view of (3), we would like this invariance to be a consequence of local super symmetry, then we shall impose the local super symmetry constraint computed above. This constraint, as explained in the lecture notes, allows to have finite fermionic mass spectrum, and the consequent local super symmetry will allow to gauge away non–physical longitudinal modes of ψ^μ.

Consider the canonical quantization of the super particle setting for simplicity $e = 1$. Show form the canonical commutation relations that $\{\psi^\mu(\tau), \psi^\nu(\tau)\} = \eta^{\mu\nu}$. **hint:** *in order to work with a well defined canonical momentum Π_ψ associated with ψ it is advisable to follow the procedure described in section 11.4 of the lecture notes.* What can you conclude about the states generated by ψ^μ?

We can construct a locally super symmetric action by introducing a new auxiliary Grassmann valued field ν to be regarded as the super partner of the auxiliary field e and modifying the action as follows:

$$S = -\frac{1}{2}\int d\tau \left(e(\dot{x}^\mu)^2 - ie\psi^\mu\dot{\psi}_\mu - 2i\nu\dot{x}^\mu\psi_\mu\right) \tag{5}$$

Show that this action is invariant under the following local super symmetry transformations:

$$\delta x^\mu = i\epsilon\psi^\mu \; ; \quad \delta\psi^\mu = \epsilon\dot{x}^\mu \tag{6}$$

$$\delta e = -2i\epsilon\nu \; ; \quad \delta\nu = \dot{\epsilon}e - \frac{1}{2}\epsilon\dot{e} \tag{7}$$

Compute the field equations for the various fields and verify that the field equation for ν yields the local super symmetry constraint derived previously. If we regard the physical states of this theory as space–time fields what condition does the local super symmetry constraint imply on these fields?

Locally super symmetric theories are called super gravity theories since they include invariance under general coordinate transformation, which is the symmetry of General Relativity, as a consequence of local super symmetry. The field e is called the *graviton* and its superpartner ν the *gravitino*. In one and two dimensions both these fields are non–propagating.

28. Consider the superstring action (eq. (11.43) of the lecture notes):

$$S = -\frac{1}{2\pi}\int d^2\sigma\,\left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{8}$$

Show that under the super symmetry trnsformations in eq. (11.42) of the lecture notes the action transforms as in (11.44). What is the constraint for *local* super symmetry?
Also in this case this constraint can be derived from a locally super symmetric action as the field equation of an auxiliary gravitino χ^α, super partner of the graviton e^a_α. The construction of this action follows the same lines as in the super symmetric particle case, i.e. with the addition of a term of the form $\overline{\chi}^\alpha J_\alpha$. In this case however the addition of a further quadratic term in the gravitino field is required by local super symmetry invariance. The derivation of this action is described in detail in section 4.3.5 of the Green–Schwarz–Witten book. It is lengthy and we shall not deal with it in this exercise. However there are some properties of Majorana spinors in two dimensions which are needed for this derivation and are useful to derive. Show that:

$$\begin{aligned}\overline{\psi}^\mu \rho^\alpha \nabla_\alpha \psi_\mu &= \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \\ \psi^\mu_A \overline{\psi}_{B\mu} &= -\frac{1}{2}\overline{\psi}_\mu \psi^\mu \,\delta_{AB}\end{aligned} \tag{9}$$

29. Given local left–moving and right–moving fields $\Phi(\tau+\sigma)$, $\tilde{\Phi}(\tau-\sigma)$ on the world sheet of a string ($\Phi \equiv \tilde{\Phi}$ for the open string), which are expanded in Fourier modes as follows:

$$\begin{aligned}\underline{\text{open string:}}\quad \Phi(\tau\pm\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-in(\tau\pm\sigma)} \\ \underline{\text{closed string:}}\quad \Phi(\tau+\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-2in(\tau+\sigma)} \\ \tilde{\Phi}(\tau-\sigma) &= \frac{1}{\Delta}\sum_n \tilde{\Phi}_n\, e^{-2in(\tau-\sigma)}\end{aligned} \tag{10}$$

where Δ is a normalization factor. Show that the following inverse relations hold:

$$\underline{\text{open string:}}\quad \Phi_n = \frac{\Delta}{2\pi}\int_0^\pi d\sigma\,\left[\Phi(\tau-\sigma)\,e^{in(\tau-\sigma)} + \Phi(\tau+\sigma)\,e^{in(\tau+\sigma)}\right]$$

$$\underline{\text{closed string:}} \qquad \Phi_n = \frac{\Delta}{\pi} \int_0^\pi d\sigma\, \Phi(\tau+\sigma)\, e^{2in(\tau+\sigma)}$$

$$\tilde{\Phi}_n = \frac{\Delta}{\pi} \int_0^\pi d\sigma\, \tilde{\Phi}(\tau-\sigma)\, e^{2in(\tau-\sigma)} \qquad (11)$$

Let Φ and $\tilde{\Phi}$ be fermionic fields with Neveu-Schwarz boundary conditions, express them as Fourier series and find the inverse transformations yielding the coefficients.

Using the above relations derive the expressions for the Fourier coefficients α_n^μ, d_n^μ, b_r^μ, L_n, F_n, G_r as functions of $\partial_+ X^\mu$, ψ_+^μ (Ramond), ψ_+^μ (Neveu–Schwarz), T_{++}, J_+ (Ramond), J_+ (Neveu–Schwarz) respectively and the corresponding left–handed quantities, for open and closed strings. Recall that:

$$\underline{\text{open string:}} \qquad \partial_\pm X^\mu = \frac{1}{2} \sum_n \alpha_n^\mu e^{-in(\tau\pm\sigma)}$$

$$\psi_\pm^\mu\,(\text{Ramond (R)}) = \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-in(\tau\pm\sigma)}$$

$$\psi_\pm^\mu\,(\text{Neveu–Schwarz (NS)}) = \frac{1}{\sqrt{2}} \sum_{r\in Z+1/2} b_r^\mu e^{-ir(\tau\pm\sigma)}$$

$$T_{\pm\pm} = \frac{1}{2} \sum_n L_n\, e^{-in(\tau\pm\sigma)}$$

$$J_\pm\,(\text{Ramond}) = \frac{1}{2\sqrt{2}} \sum_n F_n\, e^{-in(\tau\pm\sigma)}$$

$$J_\pm\,(\text{Neveu–Schwarz}) = \frac{1}{2\sqrt{2}} \sum_{r\in Z+1/2} G_r\, e^{-ir(\tau\pm\sigma)}$$

$$\underline{\text{closed string:}} \qquad \partial_+ X^\mu = \sum_n \alpha_n^\mu e^{-2in(\tau+\sigma)}$$

$$\psi_+^\mu\,(\text{Ramond}) = \sum_n d_n^\mu e^{-2in(\tau+\sigma)}$$

$$\psi_+^\mu\,(\text{Neveu–Schwarz}) = \sum_{r\in Z+1/2} b_r^\mu e^{-2ir(\tau+\sigma)}$$

$$T_{++} = 2 \sum_n L_n\, e^{-2in(\tau+\sigma)}$$

$$J_+\,(\text{Ramond}) = \sum_n F_n\, e^{-2in(\tau+\sigma)}$$

$$J_+\,(\text{Neveu–Schwarz}) = \sum_{r\in Z+1/2} G_r\, e^{-2ir(\tau+\sigma)}$$

$$
\begin{aligned}
\partial_- X^\mu &= \sum_n \tilde{\alpha}_n^\mu e^{-2in(\tau-\sigma)} \\
\psi_-^\mu \text{ (Ramond)} &= \sum_n \tilde{d}_n^\mu e^{-2in(\tau-\sigma)} \\
\psi_-^\mu \text{ (Neveu–Schwarz)} &= \sum_{r\in Z+1/2} \tilde{b}_r^\mu e^{-2ir(\tau-\sigma)} \\
T_{--} &= 2 \sum_n \tilde{L}_n \, e^{-2in(\tau-\sigma)} \\
J_- \text{ (Ramond)} &= \sum_n \tilde{F}_n \, e^{-2in(\tau-\sigma)} \\
J_- \text{ (Neveu–Schwarz)} &= \sum_{r\in Z+1/2} \tilde{G}_r \, e^{-2ir(\tau-\sigma)}
\end{aligned} \tag{12}
$$

Use the above definitions to express the constraints L_n, F_n, G_r and the corresponding tilded quantities in terms of field coefficients for both open and closed strings.

30. From the canonical commutation/anti–commutation relations:

$$
\begin{aligned}
\left[\dot{X}^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\left[X^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= \left[\dot{X}^\mu(\sigma,\tau),\, \dot{X}^\nu(\sigma',\tau)\right] = 0 \\
\{\psi_\pm^\mu(\sigma,\tau),\, \psi_\pm^\nu(\sigma',\tau)\} &= \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\{\psi_\pm^\mu(\sigma,\tau),\, \psi_\mp^\nu(\sigma',\tau)\} &= 0
\end{aligned} \tag{13}
$$

using the results and conventions of the previous exercise, deduce the commutation/anti–commutation relations among α_n^μ, d_n^μ, b_r^μ and the corresponding left–moving quantities (see eqs. (11.38), (11.39) of the lecture notes) as operators in the *unconstrained* Hilbert space. Write the constraints L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$ as normal ordered operators (recall that the definition of normal order for fermionic operators is the same as in the bosonic case except for a minus sign required every time the order of two operators is inverted: $: d_n d_{-n} := -d_{-n} d_n = - : d_{-n} d_n :$). Deduce the commutation/anti–commutation relations among L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$.

When a constraint is expressed as a quadratic function of quantities which, as operators, do not commute (bosonic operators) or anti–commute (fermionic operators), then their implementation at the quantum level suffers from an ordering ambiguity and therefore requires the introduction of a constant. For the bosonic string we saw that the only constraint with this ambiguity was L_0 and therefore a constant a was introduced so that in the light–cone quantization the constraints had the form $L_n - a\delta_n = 0$. In the superstring case, which of the constraints requires an ordering constant in the R and NS sectors? The ordering constants in the two sectors are in general unrelated. Consider the R sector and suppose we implement the constraint F_0 in the form $F_0 - \mu = 0$, μ being a c–number, find the relation between the constant a for L_0 and μ (use the relation $\{F_0, F_0\} \sim L_0$). Is it consistent to set $\mu = 0$? What consequence dose this have on the value of a for the R sector?

Let us denote by $|0\rangle_R$ and $|0\rangle_{NS}$ the vacua of the R and NS sectors (for the open string or for a single left or right mover sector of the closed string) respectively, show that $|0\rangle_{NS}$ has degeneracy one while for $D = 2n$ $|0\rangle_R$ has degeneracy 2^n and therefore describes a fermionic state (show that the action of d_0^μ on $|0\rangle_R$ does not change the energy of the state and use the anti–commutation relations among the d_0^μ.). How many sectors does the closed string have? Write the vacua for each closed string sector.

31. **Light–cone quantization:** Consider the gauge fixing conditions $X^+ = x^+ + p^+\tau$ and $\psi^+ = 0$. Show that they are consistent with super symmetry transformations (equations (11.42) of the lecture notes). Solve them in terms of the α_n^+, d_n^+, b_r^+, $\tilde{\alpha}_n^+$, $\tilde{d}_n^+$, $\tilde{b}_r^+$ in the R, NS, left and right–moving sectors. Use these conditions to solve the constraints by expressing α_n^-, d_n^-, b_r^-, $\tilde{\alpha}_n^-$, $\tilde{d}_n^-$, $\tilde{b}_r^-$ in terms of the corresponding transverse components for the relevant sectors in the open and closed string cases using $a = 0$ in the R sector (see previous exercise) and $a = 1/2$ in the NS sector (recall that $\alpha_0^\mu = p^\mu$ for the open string and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu/2$ for the closed string). Write the formula for M^2 in the various sectors of the open and closed string. Write explicitly the tachyonic and massless open and closed string states.