32. Find the expression of the vacuum expectation value $G^{F\,\mu\nu}$ of the product of two fermionic fields:

$$\langle 0|\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma')|0\rangle \;=\; G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma') \tag{1}$$

as for the bosonic case, the function G^F is the Green function of the fermionic field equation and, when time ordered, yields the fermion propagator.
Hint: *express the product of the two fields in terms of its normal ordered expression:*

$$\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma') \;=\; :\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma'): + G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma') \tag{2}$$

33. **On $SO(1,9)$ and $SO(8)$.** The groups $SO(1,9)$ and $SO(8)$ are defined as those groups of transformations whose action on the corresponding vector representations V^μ, v^i ($\mu = 0,\ldots,9,\ i = 1,\ldots,8$) leaves the metrics $\eta_{\mu\nu}$ and δ_{ij} invariant:

$$\begin{aligned} U^\mu{}_\nu &\in SO(1,9) \;\Leftrightarrow\; U^\rho{}_\mu\,\eta_{\rho\sigma}\,U^\sigma{}_\nu \,=\, \eta_{\mu\nu} && (3)\\ \tilde{U}^i{}_j &\in SO(8) \;\Leftrightarrow\; \tilde{U}^k{}_i\,\delta_{kn}\,\tilde{U}^n{}_j \,=\, \delta_{ij} && (4)\\ \eta^{\mu\nu} &= \mathrm{diag}(-,\,+,\,\ldots,\,+) \end{aligned}$$

The vector representation of $SO(8)$, described by v^i, is usually denoted by $\mathbf{8}_v$. Let us define the following matrices:

$$\begin{aligned} \mathbb{M}^\mu{}_\nu &= w_{\rho\sigma}\,(M^{\rho\sigma})^\mu{}_\nu \;;\; (M^{\rho\sigma})^\mu{}_\nu \,=\, \eta^{\sigma\mu}\,\delta^\rho_\nu - \eta^{\sigma\nu}\,\delta^\rho_\mu \\ \tilde{\mathbb{M}}^i{}_j &= w_{kn}\,(\tilde{M}^{kn})^i{}_j \;;\; (\tilde{M}^{kn})^i{}_j \,=\, \delta^{ni}\,\delta^k_j - \delta^{ki}\,\delta^n_j \end{aligned} \tag{5}$$

where $w_{\mu\nu}$ and its restriction w_{ij} to the $i,\,j$ indices are antisymmetric matrices of parameters. Show that $\mathbb{M}^\mu{}_\nu$ and $\tilde{\mathbb{M}}^i{}_j$ are infinitesimal generators of $SO(1,9)$ and $SO(8)$ respectively in their vector representations, i.e. that $U^\mu{}_\nu = \delta^\mu{}_\nu + \mathbb{M}^\mu{}_\nu$ and $\tilde{U}^i{}_j = \delta^i{}_j + \tilde{\mathbb{M}}^i{}_j$ fulfill equations (3) and (4) respectively to first order in w. How many independent generators do $SO(1,9)$ and $SO(8)$ have? Using the definitions (5) show that the $SO(1,9)$ infinitesimal generators $M^{\mu\nu}$ fulfill the following commutation relations:

$$[M^{\mu\nu},\,M^{\rho\sigma}] \;=\; \eta^{\mu\rho}\,M^{\nu\sigma} + \eta^{\nu\sigma}\,M^{\mu\rho} - \eta^{\nu\rho}\,M^{\mu\sigma} - \eta^{\mu\sigma}\,M^{\nu\rho} \tag{6}$$

and that the $SO(8)$ generators $\tilde{M}^{ij}$ fulfill relations obtained by restricting eq. (6) to the i, j indices.

Consider the $SO(1,9)$ and $SO(8)$ spinorial representations. What are the dimensions of an $SO(1,9)$ and an $SO(8)$ spinors? Let us introduce the $SO(1,9)$ and $SO(8)$ Clifford algebras Γ^μ, γ^i fulfilling $\{\Gamma^\mu,\, \Gamma^\nu\} = -2\,\eta^{\mu\nu}$ and $\{\gamma^i,\, \gamma^j\} = -2\,\delta^{ij}$ (notice here a sign difference with respect to the definition of the $SO(8)$ gamma matrices given in the lecture notes). Show that the matrices $M^{\mu\nu} = \Gamma^{\mu\nu}/2 = (\Gamma^\mu\,\Gamma^\nu - \Gamma^\nu\,\Gamma^\mu)/4$ and $\tilde{M}^{ij} = \gamma^{ij} = (\gamma^i\,\gamma^j - \gamma^j\,\gamma^i)/4$ are the generators of $SO(1,9)$ and of $SO(8)$ in the spinorial representation respectively, namely that they fulfill the corresponding commutation relations. To give an explicit representation to these matrices it is useful to introduce the tensor product notation: given two matrices $A_{n\times n} = (a_{ij})$ and $B_{m\times m} = (a_{ij})$, the matrix $C_{nm\times nm} = A_{n\times n}\otimes B_{m\times m}$ is an $nm\times nm$ which is obtained by substituting each entry b_{ij} of $B_{m\times m}$ with the $n\times n$ block $b_{ij}\,A_{n\times n}$. For example consider:

$$\tau^1\otimes\tau^2 = \begin{pmatrix} 0_{2\times 2} & -i\tau^1 \\ i\tau^1 & 0_{2\times 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

where τ^i are the Pauli matrices. This product can be iterated by defining $A\otimes B\otimes C = A\otimes(B\otimes C)$. Show that:

$$(A_1\otimes B_1)\cdot(A_2\otimes B_2) = (A_1\cdot A_2)\otimes(B_1\cdot B_2) \tag{8}$$

Show that the γ^i can be expressed as the tensor product of two by two matrices in the following way:

$$\begin{aligned}
\gamma^1 &= i\tau^1\otimes \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \\
\gamma^2 &= i\tau^2\otimes \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \\
\gamma^3 &= i\tau^3\otimes \tau^1\otimes\mathbb{1}\otimes\mathbb{1} \\
\gamma^4 &= i\tau^3\otimes \tau^2\otimes\mathbb{1}\otimes\mathbb{1} \\
\gamma^5 &= i\tau^3\otimes \tau^3\otimes\tau^1\otimes\mathbb{1} \\
\gamma^6 &= i\tau^3\otimes \tau^3\otimes\tau^2\otimes\mathbb{1} \\
\gamma^7 &= i\tau^3\otimes \tau^3\otimes\tau^3\otimes\tau^1 \\
\gamma^8 &= i\tau^3\otimes \tau^3\otimes\tau^3\otimes\tau^2
\end{aligned} \tag{9}$$

The above representation is different from the one given in the lecture notes in which all the γ^i are imaginary and the $SO(8)$ generators already appear in a block diagonal form according to the decomposition into chiral representations. In the representation (9) however the generators which we will need to diagonalize in the sequel are diagonal to start with.
Compute the matrix $\gamma = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4\,\gamma^5\,\gamma^6\,\gamma^7\,\gamma^8$ and show that it is the chirality matrix for the $SO(8)$ spinor representation, namely that $(\gamma)^2 = \mathbb{1}_{16\times 16}$ and that $\{\gamma,\,\gamma^i\} = 0$. Is γ diagonal? What are its eigenvalues? Show that the matrix $B = \tau^2 \otimes \tau^1 \otimes \tau^2 \otimes \tau^1$ fulfills $(B)^2 = \mathbb{1}_{16\times 16}$ and $B^{-1}\,\gamma^i\,B = \gamma^{i*}$. Using the results of ex. 25 and the matrices B and γ express the Majorana and the Weyl conditions on an $SO(8)$ spinor λ and show that they are compatible, so that a Majorana–Weyl spinor can be defined.

Now show that the following matrices realize the $SO(1,9)$ Clifford algebra:

$$
\begin{aligned}
\Gamma^0 &= \gamma \otimes \tau^2 \\
\Gamma^i &= \gamma^i \otimes \mathbb{1}_{2\times 2} \\
\Gamma^9 &= i\gamma \otimes \tau^1
\end{aligned}
\tag{10}
$$

Compute the chirality matrix Γ. Is Γ diagonal? What are its eigenvalues? Show that the matrix $B' = \tau^2 \otimes \tau^1 \otimes \tau^2 \otimes \tau^1 \otimes \tau^3$ fulfills $(B')^2 = \mathbb{1}_{32\times 32}$ and $B'^{-1}\,\Gamma^\mu\,B' = \Gamma^{\mu *}$. In terms of B' and Γ write the Majorana and the Weyl conditions on an $SO(1,9)$ spinor ξ and show that they are compatible, so that a Majorana–Weyl spinor can be defined. The group $SO(8)$ is contained inside $SO(1,9)$. The subset of the $SO(1,9)$ generators which are also $SO(8)$ generators are represented by the 32×32 matrices $M^{ij} = \Gamma^{ij}/2$. Write these generators in terms of the $SO(8)$ generators given by the 16×16 matrices $\tilde{M}^{ij} = \gamma^{ij}/2$ using the tensor product notation.

Just as for states in quantum mechanics, in order to describe a basis of the $SO(1,9)$ spinorial representation we need to define a *complete set* of $SO(1,9)$ generators, i.e. a maximal set of commuting generators, so that each element of the basis of can be labeled by the simultaneous eigenvalues of these operators. A complete set of $SO(1,9)$ generators consists of five elements $\{H_a\}$ $a = 0,\ldots,4$. Show that the following

$SO(1,9)$ generators commute:

$$\begin{aligned} H_0 &= \frac{1}{2}\Gamma^0\Gamma^9 \; ; \; H_1 = \frac{1}{2}\Gamma^1\Gamma^2 \; ; \; H_2 = \frac{1}{2}\Gamma^3\Gamma^4 \; ; \; H_3 = \frac{1}{2}\Gamma^5\Gamma^6 \\ H_4 &= \frac{1}{2}\Gamma^7\Gamma^8 \end{aligned} \tag{11}$$

Compute their expression as tensor product of two by two matrices and show that as 32×32 matrices they are diagonal. Show that H_0 has real eigenvalues, to be denoted by s_0, as opposite to $H_{k=1,2,3,4}$ which have imaginary eigenvalues, to be denoted by is_k, s_k being real. You may also convince yourself that this abelian set cannot be enlarged, i.e. that there is no other $SO(1,9)$ generator commuting with all of them. A basis for the $SO(1,9)$ spinorial representation can therefore be written in the form $\{|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle\}$, each element being a Majorana spinor, simultaneous eigenvector of the H_a:

$$H_a\,|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \;=\; s_a\,|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \tag{12}$$

Write the elements of this basis explicitly according to the various combinations of values s_a. Show that all these spinors have definite chirality and that

$$\Gamma\,|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \;=\; 32\, s_0 s_1 s_2 s_3 s_4\,|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \tag{13}$$

Let us apply the same procedure for constructing a basis of the $SO(8)$ spinorial representation. Show that a complete set of generators in $SO(8)$ is provided by the following four:

$$h_1 = \frac{1}{2}\gamma^1\gamma^2 \; ; \; h_2 = \frac{1}{2}\gamma^3\gamma^4 \; ; \; h_3 = \frac{1}{2}\gamma^5\gamma^6 \; ; \; h_4 = \frac{1}{2}\gamma^7\gamma^8 \tag{14}$$

compute the expression of h_k as tensor product of two by two matrices and show that as 16×16 matrices they are diagonal with imaginary eigenvalues is_k. A basis for the $SO(8)$ spinorial representation can therefore be written in the form of Majorana spinors $\{|s_1,\, s_2,\, s_3,\, s_4\rangle\}$. Write these basis explicitly with the various labels s_k. Show that all these spinors have definite chirality and that

$$\gamma\,|s_1,\, s_2,\, s_3,\, s_4\rangle \;=\; 16\, s_1 s_2 s_3 s_4\,|s_1,\, s_2,\, s_3,\, s_4\rangle \tag{15}$$

group the elements $|s_1,\, s_2,\, s_3,\, s_4\rangle$ according to their chirality and show that each chiral representation is generated by 8 elements. The representation with positive chirality is conventionally denoted by $\mathbf{8}_s$, the one with negative chirality by $\mathbf{8}_c$. Show that $H_k = h_k \otimes \mathbb{1}_{2\times 2}$ and therefore they belong to the $SO(8)$ subgroup of $SO(1,9)$. Using this property show that the spinors $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ can be grouped in $SO(8)$ representations $\mathbf{8}_s$ and $\mathbf{8}_c$. For each of these representations write the eigenvalues of Γ and of H_0 (s_0).
Hint: *prove that the $SO(1,9)$ generator H_0 commutes with the $SO(8)$ subgroup of $SO(1,9)$. Therefore H_0 has a fixed eigenvalue s_0 on each irreducible representation of $SO(8)$ contained in the $SO(1,9)$ spinorial representation, by Shur's lemma.*

Consider the vacuum of the Ramond sector $|k,\, 0\rangle_R$ with momentum k^μ. It was shown to be an $SO(1,9)$ spinor, and therefore is can be expanded in the basis $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$. Let us implement on this state the constraint $p^\mu\, \Gamma_\mu\, |k,\, 0\rangle_R = 0$. Since this state was shown to be massless $(k^\mu k_\mu = 0)$ we can consider a frame in which $k^0 = k$, $k^9 = k$ and $k^i = 0$. Find the elements of the basis $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ fulfilling in this frame the constraint $p^\mu\, \Gamma_\mu\, |k,\, 0\rangle_R = 0$. Which $SO(8)$ representations do they belong to? Label them by the s_0 eigenvalue and the $SO(1,9)$ chirality.
Hint: *write the constraint in the chosen frame as a projector on H_0 eigenspaces:*

$$p^\mu\, \Gamma_\mu \;=\; -2\, k\, \Gamma^0\, (\frac{1}{2} - H_0)$$

After imposing the GSO condition on $|k,\, 0\rangle_R$, namely the condition on the vacuum to have a definite $SO(1,9)$ chirality, write the spinors $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ contributing to this state for the two different eigenvalues of Γ. Which $SO(8)$ representations do they belong to?