34. In the present exercise we shall show that the world sheet of a propagating closed string is conformally equivalent to a sphere with the asymptotic states coinciding with the two polar points.

Consider a closed string propagating in space–time. Represent its world–sheet as a cylinder parametrized by the complex coordinate $w = i\sigma_2 + \sigma_1$, where as usual σ_2 is the Euclidean time coordinate which runs from $-\infty$ (incoming string) to $+\infty$ (outgoing string) and σ_1 the angular coordinate in the interval $(0, \pi)$. Since we are dealing with a closed string the points with $\sigma_1 = 0$ and $\sigma_1 = \pi$ are identified. In the conformal gauge the metric in the coordinates w, $\bar{w}$ can be written as:

$$ds^2 = e^{\phi}\, dw\, d\bar{w} \tag{1}$$

write the metric in the coordinates $z = e^{2iw}$, $\bar{z}$. Represent in these coordinates the points corresponding to the asymptotic incoming and outgoing string. Now let us make a suitable choice for the conformal factor, namely:

$$e^{\phi} = 16\, \frac{|z|^2}{(1+|z|^2)^2} \tag{2}$$

show that the corresponding ds^2 is the metric on a sphere described by stereographic coordinates and that the asymptotic states coincide with the two polar points. **hint:** *perform the change of variables* $z, \bar{z} \to \theta, \varphi$ *where* $z = \cot(\frac{\theta}{2})\, e^{i\varphi}$

In general, for a more complicated process involving several incoming and outgoing strings, it can be shown that, at tree level (that is if the surface described by the interacting strings has no holes), the conformal factor of the metric can be chosen so as to map the surface of the initial diagram into a sphere in which the asymptotic states are represented by points or *punctures*, where the corresponding vertex operators are inserted. Analogously higher loop diagrams will be mapped into two dimensional surfaces with a certain number of holes (counted by the genus g of the surface) and punctures corresponding to the asymptotic states.

35. Consider type IIA and IIB superstring theories. Write the explicit state realization of the NS–NS massless modes, namely of the scalar field ϕ,

the ten dimensional metric G_{ij} and the two form B_{ij} (also called Kalb–Ramond field). As far as the R–R sector is concerned, the zero mode states are expressed as tensor product of the left and right mover sector ground states (we shall use the notation of the lecture notes and denote by $|\psi_L\rangle$, $|\psi_R\rangle$ the R ground states with positive and negative $SO(8)$ chirality respectively (which we have called $\mathbf{8}_s$, $\mathbf{8}_c$ in ex. 33). The states corresponding to the left or right moving sector are distinguished only by their positions (to the left or to the right) in the tensor products). For type IIA and IIB the zero modes of the R–R sector are $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$. These states have to be considered as 8×8 matrices in the spinor space and therefore can be decomposed in a complete basis of generators of this space, namely $\Gamma^{i_1 i_2 \ldots i_k}$ for $k = 1, \ldots, 8$. Show that only the matrices $\Gamma^{i_1 i_2 \ldots i_k}$ for $k = 1, \ldots, 4$ contribute to the expansions of $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$.

As a first exercise let us consider the simpler case of bi–spinors ξ^α, χ^β. Show that, if we denote by $M^{\alpha\beta} = \xi^\alpha \chi^\beta$, the following expansion is true:

$$\begin{aligned} M &= \sum_i c_i \, \mathrm{Tr}(M^T \tau^i)\, \tau^i \\ c_i &= \frac{1}{\mathrm{Tr}(\tau^{iT} \tau^i)} \end{aligned} \tag{3}$$

where τ^i are the Pauli matrices and $\mathrm{Tr}(M^T \tau^i) = \xi^T \tau^i \chi$.

Similarly we can write:

$$\begin{aligned} |\psi_L\rangle\langle\psi_R| &= \sum_k c_k \, C^A_{i_1 \ldots i_k} \, \Gamma^{i_1 i_2 \ldots i_k} \\ |\psi_L\rangle\langle\psi_L| &= \sum_k c_k \, C^B_{i_1 \ldots i_k} \, \Gamma^{i_1 i_2 \ldots i_k} \\ C^A_{i_1 \ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2 \ldots i_k}|\psi_R\rangle \\ C^B_{i_1 \ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2 \ldots i_k}|\psi_L\rangle \end{aligned} \tag{4}$$

where c_k are numerical constants. Show that the fields $C^A_{i_1 \ldots i_k}$ are non vanishing only for $k = 1, 3$ and $C^B_{i_1 \ldots i_k}$ are non vanishing only for $k = 0, 2, 4$. Which $SO(8)$ representations do the C^A and C^B tensors belong to?

Write the explicit state realization of the NS–R and R–NS massless modes. Perform the counting of fermionic and bosonic degrees of freedom.

36. In the present exercise we shall study the unoriented closed superstring theory. Unoriented theories are invariant under the transformation $\Omega : \sigma \to \pi - \sigma$. What is the effect of this transformation on the left and right mode operators? Show that only type IIB theory can be made invariant under Ω. Show that the NS–NS fields ϕ, G_{ij} are even under Ω and that B_{ij} is odd **hint:** *write the corresponding states as components of $b^i_{-1/2}\,\tilde{b}^j_{-1/2}\,|0,0\rangle$ and show that the action of Ω amounts to $i \leftrightarrow j$.*

    As far as the R–R sector is concerned, show that under Ω C_{ij} is even and C_{ijkl} is odd. As far as the fermion fields are concerned show that the effect of Ω is to switch NS-R $\leftrightarrow$ R–NS and therefore only a symmetric combination of the two survives (is even). Perform a counting of the fermionic and bosonic degrees of freedom and motivate the fact that the amount of supersymmetry is $\mathcal{N} = 1$.