Exam rehearsal

April 27, 2002

1. Consider a space with the metric $g_{\mu\nu}(x)$ and the Levi-Civita Christoffel symbols $\Gamma^{\rho}_{\mu\nu}(x)$.

   (a) Use the fact, that the metric is covariantly constant, i.e.

   $$D_\mu g_{\alpha\beta} = 0 \tag{1}$$

   to solve for the Christoffel symbols in terms of the metric.

   (b) Consider a curve $x^\mu(s)$ with affine parameter s. The geodesic equation for this curve is given by

   $$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{2}$$

   where the dot indicates differentiation with respect to s. Show that the geodesic equation (2) follows from the Euler-Lagrange equations corresponding to the Lagrangian

   $$L = \frac{1}{2}\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}(x) \tag{3}$$

   (c) Consider a metric with a 2-sphere S^2 with coordiantes $(r,\, \phi)$:

   $$ds^2 = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\phi^2). \tag{4}$$

   Use the result of (b) to calculate the Christoffel symbols corresponding to the metric given in (4).

2. Consider the Robertson-Walker metric for $k = 1$.

   $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\,\phi^2)\right] \tag{5}$$

   For the stress-energy tensor of a perfect fluid the Einstein equations lead to the following relations between the function $a(t)$, the mass density $\rho(t)$ and the pressure $p(t)$:

   $$\frac{\dot{a}^2 + 1}{a^2} = \frac{1}{3}\kappa\rho \tag{6}$$

   $$\dot{\rho} + 3(p+\rho)\frac{\dot{a}}{a} = 0 \tag{7}$$

(a) Show that (7) comes from the Bianchi identity $D_\mu T^{\mu\nu} = 0$ with stress-energy of the perfect fluid:

$$T^{\mu\nu} = -(p+\rho)u^\mu u^\nu - pg^{\mu\nu} \tag{8}$$

where $u^\mu = dx^\mu/d\tau$ is the four-velocity of the fluid. Use the comoving frame (i.e. $u^\mu = (-1,\, 0,\, 0,\, 0)$).

(b) From now on consider the situation of a Friedmann universe with non-relativistic matter, i.e. $p = 0$. Show that ρa^3 is constant.

(c) Show that $a(t)$ satisfies the differential equation

$$\left(\frac{da}{dt}\right)^2 + 1 = \frac{A^2}{a} \tag{9}$$

where A is a constant. We impose the boundary condition that $a(t = 0) = 0$. Show that, given these boundary conditions, the solution of the differential equation (9) is given by the equations

$$a = \frac{A^2}{2}(1 - \cos\psi) \tag{10}$$

$$t = \frac{A^2}{2}(\psi - \sin\psi) \tag{11}$$

where ψ is a parameter. Draw the graph of the function $a(t)$. Derive an expression for the time $t_0 > 0$ in trems of the constant A at which for the first time after $t = 0$ the radius a becomes zero, i.e. $a(t_0) = 0$.

3. Two stars of approximately equal masses move on circular orbits around each other. Assume that their interaction is approximately described by Newton's law, assuming that their velocities are not too great.

(a) Write down the mass distribution as a function of time and calculate the quadrupole moment of this system.

$$Q_{ij} = \int (x^i x^j - \tfrac{1}{3}\, x^2 \delta_{ij}) T^{00} d^3x \tag{12}$$

(b) Find the energy loss (i.e. the emitted enerrgy per unit time) for this system due to gravitational radiation.

(c) The energy loss leads to a gradual approach of the orbits (the so called secular approach). Find the speed of this approach.

4. Show that $\sqrt{-\det g}\, d^4x$ is a covariant quantity, that is it does not change under coordinate transformations. (That is $S = \int d^4x \sqrt{-\det g}\, (R - 2\Lambda)$ is well defined action for Einstein's theory of General Relativity).

If one wants to check one's speed, three hours ought to be enough to solve these problems. Note this is just an example set of exercises, Schwarzschild geometry, basic quantities like metric, connection field, curvature, Maxwell field-strength, stress-energy, their symmetry properties, gravitational radiation and variation principle are all likely to appear in the exam. Solving the werkcollege exercises provides a great help. The lecture notes may be used during the exam. The date of the exam is 23-05-2002, 14:00, the location and any new information is to be found at http://www.phys.uu.nl/~thooft/lectures/gr.html as well as the solutions of the werkcollege exercises and the lecture notes.