Cohomological Finite Generation

Wilberd van der Kallen

Universiteit Utrecht

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P. Gordan 1868,
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Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.

In modern language: $G = SL_2(\mathbb{C})$ as algebraic group.
$G \curvearrowright V := \mathbb{C}^2, \quad \mathbb{C}[V] = \mathbb{C}[X,Y],$
$W_d := \mathbb{C}[V]_d,\ W_2 = \{aX^2 + bXY + cY^2\}, \quad \mathbb{C}[W_2] = \mathbb{C}[a,b,c],$
$b^2 - 4ac \in \mathbb{C}[W_2]^G$ an *invariant* (= fixed point).

Gordan: $\mathbb{C}[W_d]^G$ is finitely generated (f.g.) as a $\mathbb{C}$-algebra.

Hilbert 1890

$G = SL_n(\mathbb{C})$ acting algebraically on some finite dimensional complex vector space V.

Here 'algebraically' means the action is given by polynomials: For each $v \in V$ there is a polynomial f_v in the matrix entries of $g \in G$ with coefficients in V so that $g \cdot v = f_v(g)$.

Example: The above action of $G = SL_2(\mathbb{C})$ on W_2.

Then Hilbert shows nonconstructively ('theology') that $\mathbb{C}[V]^G$ is finitely generated as a $\mathbb{C}$-algebra.

Examples Consider the action of $G = GL_n(\mathbb{C})$ by conjugation on the vector space $V = M_n(\mathbb{C})$ of $n \times n$ matrices. So $g \in G$ sends $m \in V$ to gmg^{-1}. Then $\mathbb{C}[V]^G$ is generated by the coefficients c_i of the characteristic polynomial $\det(m - \lambda I) = c_0 + c_1\lambda + \cdots c_n\lambda^n$.

Next let G be the group of permutations of the n variables in the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. Then $\mathbb{C}[X_1, \ldots, X_n]^G = \mathbb{C}[p_1, \ldots, p_n]$, where $p_i = X_1^i + \cdots X_n^i$.

Encouraged by an incorrect claim of Maurer Hilbert asked in his fourteenth problem if this finite generation of invariants is a general fact about actions of algebraic Lie groups on domains of finite type over $\mathbb{C}$.

A counterexample of Nagata (1959) showed this was too optimistic.

By then it was understood that finite generation of invariants holds for compact connected real Lie groups like orthogonal groups (cf. Hurwitz 1897). Hurwitz considers compact group K with Haar measure dk and introduces the method of averaging. $K \curvearrowright V$ linear. Get linear equivariant retract $V \to V^K$ from

$$v \longmapsto \frac{\int_K kv\ dk}{\int_K dk}.$$

Finite generation also holds for the complexifications of compact Lie groups, also known as the connected reductive complex algebraic Lie groups (Weyl 1926).

Finite groups have been treated by Emmy Noether (1926), so connectedness may be dropped. (Algebraic Lie groups have finitely many connected components.)

Mumford (1965) needed finite generation of invariants for reductive algebraic groups over fields of arbitrary characteristic in order to construct moduli spaces.

Say k is an infinite field and $G = SL_n(k)$ is acting algebraically on some finite dimensional k-vector space V.
Then Mumford needs in particular that $k[V]^G$ is finitely generated as a k-algebra.

In his book Geometric Invariant Theory (1965) Mumford introduced a condition, often referred to as *geometric reductivity*. He conjectured it to be true for reductive algebraic groups and he conjectured it implies finite generation of invariants.
These conjectures were confirmed by Haboush (1975) and Nagata (1964) respectively.
Nagata treated any algebra of finite type over the base field, not just domains. We adopt this generality. It rather changes the problem of finite generation of invariants.

The proof of Nagata was actually based on a property that Franjou and the speaker call 'power reductivity'.

We call G *power reductive* if, whenever G acts algebraically on a commutative k-algebra A, leaving invariant an ideal I, there is for every $f \in (A/I)^G$ a power f^n that lifts to A^G.

For a group like $SL_m(\mathbb{C})$ one could take $n = 1$.
Actually Mumford wrote his book after this work of Nagata, and his formulation of the problem amounts to conjecturing power reductivity for the relevant G. As long as the base ring is a field, power reductivity is easily equivalent to what is known as geometric reductivity, which we do not define here.

The standard story has the Mumford conjecture stated in terms of geometric reductivity.
We have followed that custom here, even though we know that over more general base ring power reductivity is the superior notion.

Let us say that G satisfies property (FG) if, whenever G acts on a commutative algebra of A finite type over k, the ring of invariants A^G is also finitely generated over k. So then the theorem of Haboush and Nagata says that connected reductive algebraic groups like SL_n over a field have property (FG). Of course the action of G on A should be consistent with the nature of G and A respectively. Thus if G is an algebraic group, then the action should be algebraic and the multiplication map $A \otimes_k A \to A$ should be equivariant.

It turns out that (FG) is equivalent to power reductivity, and this continues to hold if our field k is replaced with an arbitrary commutative noetherian base ring. (The correct formulation now involves group schemes, not groups.) Here it is essential that we changed the rules by allowing any algebra of finite type over the base ring, not just domains.

This equivalence may be used to prove that connected reductive group schemes like SL_m have property (FG) over an arbitrary commutative noetherian base ring R.

Example

Let $R = \mathbb{Z}$, $G = SL_2$ acting in its adjoint representation M with basis

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So $g \in G$ sends a matrix m to gmg^{-1}. The class of H in $M/2M$ is invariant. It is very common that such a modular invariant does not lift to characteristic zero.

But when we embed M into an algebra we can apply Power Reductivity. It tells that actually some power of

$H \bmod 2M \in (S^*(M/2M))^G$ must lift to $(S^*M)^G$.

Indeed $H^2 + 4XY \in (S^2M)^G$.

Note that it is essential that one allows nonfree modules when forging this link between characteristics.

The equivalence of (FG) with power reductivity leads to easy counterexamples to (FG). For instance, let G be the Lie group $\mathbb{C}$ with addition as operation.

Let $t \in G$ act on $A = \mathbb{C}[X, Y, Z]/(XZ)$ by
$X \mapsto X, \quad Y \mapsto Y + tX, \quad Z \mapsto Z$.
Then A^G contains X, Z and $Y^i Z$ for $i \geq 1$, and A^G is not finitely generated. This is an awful lot simpler than the famous Nagata counterexample from 1959. But we have changed the rules and A is no domain.

If I is the ideal generated by X in A, then one also sees that G fails power reductivity. No power of $Y \in (A/I)^G$ lifts to A^G.

Such failure of lifting is what the cohomology group $H^1(G, I)$ is about. So we naturally end up studying cohomology when looking at invariant theory. One has $A^G = H^0(G, A)$ and the $H^i(G, -)$ are the derived functors of the fixed point functor $(-)^G$.

Let us say that G satisfies the cohomological finite generation property (CFG) if, whenever G acts on a commutative algebra A of finite type over k, the cohomology algebra $H^*(G, A)$ is also finitely generated over k.

Evens (1961) proved that finite groups have (CFG) and this has been the starting point of the theory of *support varieties*. In this theory one exploits a connection between the rate of growth of a minimal projective resolution and the dimension of a 'support variety', which is a subvariety of the spectrum of $H^{\text{even}}(G, k)$.

People working in representation theory of algebraic groups wanted to join this activity. Thus one needed to show that the result of Evens extends to more general finite group schemes. (An algebraic group scheme G is called finite if its coordinate ring $k[G]$ is a finite dimensional vector space.) This turned out to be *surprisingly elusive* (Friedlander Suslin 1997).

Friedlander and Suslin had to invent a new representation theory, the *strict polynomial functors*, in order to construct universal cohomology classes that enabled them to bring some Hochschild–Serre spectral sequences under control.

Their representation theory uses the Schur algebras $S(n, d)$ introduced by I. Schur in his 1901 thesis.

The $S(n,d)$-modules correspond with polynomial representations, homogeneous of degree d, of GL_n. The setting of Friedlander and Suslin captures $S(n,d)$ for all n simultaneously. Intuitively one thus finds the behaviour as $n \to \infty$.

Now I had noticed that if one could show that GL_n has (CFG) for large n, then it would follow that finite group schemes have (CFG). I could soon prove (2004) that GL_2 has (CFG), but 2 is not large. Then I started to find corollaries to (CFG) that seemed wrong. So the game became to disprove the corollaries. This was a big failure. Instead of disproving I started to prove more and more cases. Thus it became my *conjecture* that GL_n has (CFG) (when the base ring is a field).

To follow the strategy of Friedlander and Suslin and prove my conjecture, more universal cohomology classes were needed.

This required a two variable variant of strict polynomial functors, the *strict polynomial bifunctors* of Franjou and Friedlander (2008), and some miraculous arguments of Touzé (2010).

One now wonders if (CFG) still holds for GL_n when the base ring is just a commutative noetherian ring R. It is so for $n = 2$ and also if R contains a field.

We are not aware of striking applications of the general (CFG) theorem, but investigating the (CFG) conjecture has led to new insights. As $H^{>0}(G, k)$ vanishes for reductive G, there is no obvious theory of support varieties for reductive G.