Reductivity and finite generation

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The group scheme GL_N

The group scheme GL_N over the base ring $\mathbf{k}$ associates to a commutative $\mathbf{k}$-algebra R the group $GL_N(R)$ of invertible N by N matrices with entries in R.
- Its coordinate ring $\mathbf{k}[GL_N]$ is generated over $\mathbf{k}$ by the x_{ij} and $1/\det$, where x_{ij} picks out the ij-th matrix entry. Every $f \in \mathbf{k}[GL_N]$ defines a map $GL_N(R) \to R$, for every R.
- A GL_N-module is a $\mathbf{k}$-module M together with a functorial action of $GL_N(R)$ on $R \otimes_{\mathbf{k}} M$ for all commutative $\mathbf{k}$-algebras R.
- For every $v \in M$ there are $f_i \in \mathbf{k}[GL_N]$ and $m_i \in M$ such that $g \cdot v = \sum_i f_i(g) m_i$ for $g \in GL_N(\mathbf{k})$, and similarly $g \cdot (1 \otimes v) = \sum_i f_i(g)(1 \otimes m_i)$ for $g \in GL_N(R)$.
(Finite sums. This is algebra, not analysis.)

Examples

- $M = \mathbf{k}[GL_N]$ with $(g \cdot f)(x) = f(g^{-1}x)$.
- $\mathbf{k} = \mathbb{Z}$, $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^N, \mathbb{Z}^N \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \wedge^3(\mathbb{Z}^N)$.

Additive group scheme $\mathbb{G}_a$

The additive group scheme $\mathbb{G}_a$ over the base ring **k** represents the functor that associates to a commutative **k**-algebra R its additive group $\mathbb{G}_a(R) := (R, +)$.
Its behaviour is rather different than that of GL_N.
- For instance, when $\mathbf{k} = \mathbb{C}$ we may let $\mathbb{G}_a(\mathbb{C})$ act on the algebra $A = \mathbb{C}[x^2, x^3, y, z]$ such that $t \in \mathbb{C}$ sends x^i to x^i, y to $y + tx^3$, z to $z + tx^2$. The ring of invariants $A^{\mathbb{G}_a}$ has as minimal generating set over $\mathbb{C}$ the set $\{x^\alpha(y - xz)^\beta\}_{\alpha\in\{2,3\},\beta\geq 0}\}$. So $A^{\mathbb{G}_a}$ is not finitely generated (Neena Gupta).
Maurer 1899: $\mathbb{G}_a \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$ ✓
Nagata counterexample 1958: ${\mathbb{G}_a}^{13} \curvearrowright \mathbb{C}[x_1, \ldots, x_{32}]$
Totaro 2008: ${\mathbb{G}_a}^3 \curvearrowright F[x_1, \ldots, x_{18}]$
- The only irreducible representation of $\mathbb{G}_a$ over $\mathbb{C}$ is the trivial representation $\mathbb{C}$, so most representations of $\mathbb{G}_a$ are not completely reducible.

Power reductivity, a basic notion

Let G be a flat affine group scheme over $\mathbf{k}$.

Definition (after Mumford, GIT book 1965)

The group G is *power reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity)

Let $\varphi : M \twoheadrightarrow \mathbf{k}$ be a surjective map of G-modules. Then there is a positive integer d such that the d-th symmetric power of φ is a split surjection of G-modules

$$S^d\phi : S^d M \overset{\curvearrowleft}{\twoheadrightarrow} S^d\mathbf{k}.$$

In other words, one requires the kernel of $S^d\phi$ to have a complement in $S^d M$.

Equivalent formulations

Definition

A morphism of $\mathbf{k}$-algebras $\phi : S \to R$ is *power surjective* if for every element r of R there is a positive integer n such that the power r^n lies in the image of ϕ.

Proposition

Let G be a flat affine group scheme over $\mathbf{k}$. The following are equivalent

1. *G is power reductive,*
2. *For every power surjective G-homomorphism of commutative $\mathbf{k}$-algebras $f : A \to B$ the map $A^G \to B^G$ is power surjective,*
3. *For every surjective G-homomorphism of commutative $\mathbf{k}$-algebras $f : A \twoheadrightarrow B$ the ring B^G is integral over the image $f(A^G)$ of A^G.*

Hilbert's fourteenth problem

Theorem (Hilbert's fourteenth problem)

Let **k** *be a noetherian ring and let G be a flat affine group scheme over* **k**. *Let A be a finitely generated commutative* **k**-*algebra on which G acts through algebra automorphisms. If G is power reductive, then the subring of invariants A^G is a finitely generated* **k**-*algebra.*

Gordan 1868, Hilbert 1890, Hurwitz 1897, Weyl 1926, Emmy Noether 1926, Nagata 1964, Mumford 1965, Haboush 1975, Springer 1977, Seshadri 1977, Thomason 1987, Franjou–vdK 2010, Jarod Alper 2014.

Power reductivity is necessary

The theorem has a converse showing that power reductivity is necessary if one seeks finite generation of invariants in the present setting, where algebras only need to be finitely generated.
(In ancient Invariant Theory one only considers invariants in a polynomial ring over $\mathbb{C}$ with a G-action that preserves the grading.)

Proposition

Let $\mathbf{k}$ *be a noetherian ring and let* G *be a flat affine group scheme over* $\mathbf{k}$*.*
Assume that the $\mathbf{k}$*-algebra* A^G *is finitely generated for every finitely generated commutative* $\mathbf{k}$*-algebra* A *on which* G *acts through algebra automorphisms. Then* G *is power reductive.*

Base change properties

Power reductivity has marvelous base change properties.

Proposition

Let $\mathbf{k} \to S$ be a map of commutative rings.

1. *If G is power reductive, then so is G_S.*
2. *If $\mathbf{k} \to S$ is faithfully flat and G_S is power reductive, then so is G.*
3. *If $G_{\mathbf{k}_\mathfrak{m}}$ is power reductive for every maximal ideal $\mathfrak{m}$ of $\mathbf{k}$, then G is power reductive.*

Geometric reductivity

Proposition (Popov, Waterhouse)

Let **k** *be a field and let G be an affine algebraic group over* **k**. *The following are equivalent.*

1. *G is power reductive.*
2. *G is geometrically reductive.*
3. *The connected component G^o_{red} of its reduced subgroup G_{red} is reductive.*

Examples

• Recall that a finite group scheme G over a commutative ring **k** is an affine group scheme over **k** whose coordinate ring is a finitely generated projective **k**-module. They are power reductive.
• Reductive group schemes in the sense of SGA3 are also power reductive. Key case: Chevalley group G over $\mathbb{Z}_{(p)}$.

Cohomological Finite Generation

Theorem (Cohomological Finite Generation)

Let **k** *be a noetherian ring and let* G *be a reductive group scheme over* **k** *in the sense of SGA3. Let* A *be a finitely generated commutative* **k***-algebra on which* G *acts through algebra automorphisms. If* **k** *contains* $\mathbb{Z}$*, assume it actually contains* $\mathbb{Q}$*. Then* $H^*(G, A)$ *is a finitely generated* **k***-algebra.*

We do not explain the proof. According to [Laurent Fargues and Peter Scholze 2021 arXiv2102.13459] the result is deep.

Latest

Theorem (arXiv:2212.14600)

Let $\mathbf{k}$ *be a noetherian ring and let* G *be a finite group scheme over* $\mathbf{k}$*. (So it is flat and affine). Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts through algebra automorphisms.*
Then $H^*(G, A)$ *is a finitely generated* $\mathbf{k}$*-algebra.*

Evens 1961, Friedlander–Suslin 1997, Srinivas–vdK 2009, Franjou–vdK 2010, Touzé–vdK 2010, vdK 2015, vdK2022.

Sabbatical at Bielefeld

In 2007 my work with Srinivas was finished in Bielefeld. We studied $H^*(G,M)$ when $\mathbf{k}=\bar{\mathbb{F}}_p$, $G=GL_N$ and M is a finitely generated module over rings like $A=\bigoplus_{n\geq 0}\Gamma(G/B,\mathcal{L}^n)$.

Evaluating at $w_0B\in (G/B)^{B^-}$ defines a map from $\nabla(\lambda):=\Gamma(G/B,\mathcal{L})$ to its highest weight space $\mathbf{k}_\lambda$. Then $\nabla(\lambda)=\mathrm{ind}_{B^-}^G\,\mathbf{k}_\lambda$. For $v\in\mathbf{k}_\lambda$, $\mathrm{diag}(t_1,\cdots,t_N)\cdot v=t_1^{\lambda_1}\cdots t_N^{\lambda_N}v$.

Grosshans attaches to a module M a direct sum $\mathrm{ind}_{B^-}^G\,\mathrm{inf}_T^{B^-}\,M^U$ of costandard modules $\nabla(\lambda)$.

Let ht be the sum of the positive coroots: $\mathrm{ht}(\lambda)=2\sum_{i<j}\lambda_i-\lambda_j$.

Let $M_{\leq i}$ be the largest G-submodule all whose weight spaces have weights μ with $\mathrm{ht}(\mu)\leq i$. Grosshans shows that $\mathrm{gr}\,M=\bigoplus_i M_{\leq i}/M_{\leq i-1}\subseteq \mathrm{hull}_\nabla\,\mathrm{gr}\,M:=\mathrm{ind}_{B^-}^G\,\mathrm{inf}_T^{B^-}\,M^U$ as a submodule with the same U-invariants. M is said to have good filtration if $\mathrm{gr}\,M=\mathrm{hull}_\nabla\,\mathrm{gr}\,M$.

Let $S := \mathrm{gr}\, A$ be embedded into $R := \mathrm{hull}_\nabla \mathrm{gr}\, A$ for some finitely generated $\bar{\mathbb{F}}_p$-algebra A with G action. Now S is an invariant subalgebra of R with the same U-invariants. Mathieu shows that every element x of R has a power x^{p^r} in the subalgebra S.
The critical case is $R = \bigoplus_{n \geq 0} \Gamma(GL_N/B, \mathcal{L}^n)$ with a graded subalgebra S generated in degree one. He maps at $G/P = \mathrm{Proj}(R)$ to $G/\tilde{P} = \mathrm{Proj}(S)$ and notes that it is a homeomorphism.
All this makes us interested in $H^*(G, M)$ when $M := \mathrm{gr}\, A$ is sandwiched between $R := \mathrm{hull}_\nabla \mathrm{gr}\, A$ and $R^{(r)} := R^{p^r}$. By Friedlander–Suslin $H^*(G_r, M)^{(-r)}$ is a finitely generated module over an algebra with good filtration.
We get involved with spectral sequences

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \mathrm{gr}\, A)) \Rightarrow H^{i+j}(G, \mathrm{gr}\, A).$$

$$E_1^{ij} = H^{i+j}(G, \mathrm{gr}_{-i}\, A) \Rightarrow H^{i+j}(G, A).$$

The End

THANK YOU!