Reductivity and finite generation

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Invariants

P. Gordan 1868, J.f.d. reine u. angew. Math., 69 Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.
In modern language: $G = SL_2(\mathbb{C})$ as algebraic group.
$G \curvearrowright V := \mathbb{C}^2, \quad \mathbb{C}[V] = \mathbb{C}[X, Y], \quad W_d := \mathbb{C}[V]_d,$
$W_2 = \{aX^2 + bXY + cY^2\}, \quad \mathbb{C}[W_2] = \mathbb{C}[a, b, c],$
$b^2 - 4ac \in \mathbb{C}[W_2]^G$ an *invariant* (= fixed point).
$\mathbb{C}[W_d]^G$ is finitely generated (f.g.) as a $\mathbb{C}$-algebra and
$(W_d \otimes \mathbb{C}[W_d])^G$ is a noetherian $\mathbb{C}[W_d]^G$-module.

Hilbert 1890

Let $G = SL_n(\mathbb{C})$ act algebraically on some finite dimensional complex vector space V. Here 'algebraically' means the action is 'given by polynomials': For each $v \in V$ there is $f_v \in V \otimes_{\mathbb{C}} \mathbb{C}[G]$ so that $g \cdot v = f_v(g)$. Here $\mathbb{C}[G]$ denotes the coordinate ring of G, a polynomial ring in the matrix entries of $g \in G$. All actions will be algebraic. (comodules).

Example: The above action of $G = SL_2(\mathbb{C})$ on W_2. Then Hilbert shows nonconstructively ('theology') that $\mathbb{C}[V]^G$ is finitely generated as a $\mathbb{C}$-algebra.

Examples

Consider the action of $G = GL_n(\mathbb{C})$ by conjugation on the vector space $V = M_n(\mathbb{C})$ of $n \times n$ matrices. So $g \in G$ sends $m \in V$ to gmg^{-1}. Then $\mathbb{C}[V]^G$ is generated by the coefficients c_i of the characteristic polynomial $\det(m - \lambda I) = c_0 + c_1\lambda + \cdots c_n\lambda^n$.
Next let G be the group of permutations of the n variables in the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. Then $\mathbb{C}[X_1, \ldots, X_n]^G = \mathbb{C}[p_1, \ldots, p_n]$, where $p_i = X_1^i + \cdots X_n^i$.
Encouraged by an incorrect claim of Maurer Hilbert asked in his fourteenth problem if this finite generation of invariants is a general fact about actions of algebraic Lie groups on domains of finite type over $\mathbb{C}$.
A counterexample of Nagata (1959) showed this was too optimistic.

Nowadays

Nowadays we know more generally: Let G be a reductive algebraic group over a noetherian ring $\mathbf{k}$.
Modern version: If A is a f.g. $\mathbf{k}$-algebra, $G \curvearrowright A$, then A^G is f.g.
Consequently, if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module.
Traditional case: $A = \mathbb{C}[V]$, $\quad M = W \otimes A$, $G \curvearrowright V, W$ linear, $\dim V < \infty$, $\dim W < \infty$. Then M^G is noetherian over the f.g. A^G. Observe that the algebra A is isomorphic to a polynomial ring and that the action respects the grading. Note further that $M \subseteq \mathbb{C}[V \oplus W^\#]$.
The traditional case is different: Finite generation of invariants even holds when G is the additive subgroup $\mathbb{G}_a$ of $SL_2(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, acting on a polynomial algebra, respecting the grading. (Weitzenböck 1932 at Amsterdam = Maurer 1899)

How we got there

Hilbert 1890 Math. Annalen 36
Finite generation (the traditional case) for $G = SL_n(\mathbb{C})$, using the Ω process of Cayley and noetherian arguments. Nonconstructive.
Hurwitz 1897 considers compact group K with Haar measure dk and replaces the Ω process with the method of averaging. $K \curvearrowright V$ linear. Get linear equivariant retract $V \to V^K$ from

$$v \mapsto \frac{\int_K kv\, dk}{\int_K dk}.$$

Weyl 1926 takes in a semi-simple complex Lie group G a maximal compact subgroup K and notes that the retract is also G-equivariant from V to V^G.
One can then handle any base field of characteristic zero. (faithfully flat base change).

Characteristic $p > 0$

Let $\mathbf{k}$ be a field of characteristic p.
E. Noether 1926 considers a finite group G. If A is f.g. $\mathbf{k}$-algebra, $G \curvearrowright A$, then A^G is f.g. Also if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module. Further A is integral over A^G.
Mumford GIT 1965. Say reductive G acts on the affine variety $\mathrm{Spec}(A)$. One wants to form the quotient $\mathrm{Spec}(A)/G$, hopes it is affine and in fact equal to $\mathrm{Spec}(A^G)$. In particular one wants again that A^G is f.g. But equivariant linear retracts are no longer available.
Consider $G \curvearrowright M$, $\dim M < \infty$, $f : M \twoheadrightarrow \mathbf{k}$. Then Mumford asks for an $n = p^\nu$ so that the kernel of $S^n f$ has a complement.
Nagata has shown that a positive solution implies finite generation of A^G.

Power reductivity, a basic notion

Let G be a flat affine group scheme over $\mathbf{k}$.

Definition (after Mumford, GIT book 1965)

The group G is *power reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity)

Let $\varphi : M \twoheadrightarrow \mathbf{k}$ be a surjective map of G-modules. Then there is a positive integer d such that the d-th symmetric power of φ is a split surjection of G-modules

$$S^d\phi : S^d M \overset{\curvearrowleft}{\twoheadrightarrow} S^d\mathbf{k}.$$

In other words, one requires the kernel of $S^d\phi$ to have a complement in $S^d M$.

Remark

If no such d exists, then $(S^*\mathbf{k})^G$ is no noetherian $(S^*M)^G$ module.

Equivalent formulations

Definition

A morphism of $\mathbf{k}$-algebras $\phi : S \to R$ is *power surjective* if for every element r of R there is a positive integer n such that the power r^n lies in the image of ϕ.

Proposition

Let G be a flat affine group scheme over $\mathbf{k}$. The following are equivalent

1. *G is power reductive,*
2. *For every power surjective G-homomorphism of commutative $\mathbf{k}$-algebras $f : A \to B$ the map $A^G \to B^G$ is power surjective,*
3. *For every surjective G-homomorphism of commutative $\mathbf{k}$-algebras $f : A \twoheadrightarrow B$ the ring B^G is integral over the image $f(A^G)$ of A^G.*

Example

Let $\mathbf{k} = \mathbb{Z}$, $G = SL_2$ acting in its adjoint representation M with basis $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The class of H in $M/2M$ is invariant. One does not expect such a modular invariant to lift to characteristic zero.
However, G is power reductive, and this tells that actually some power of H mod $2M \in (S^*(M/2M))^G$ must lift to $(S^*M)^G$.
Indeed $H^2 + 4XY \in (S^2M)^G$.
Note that it is essential that one allows nonfree modules when forging this link between characteristics.

Theorem (Hilbert's fourteenth problem)

Let $\mathbf{k}$ *be a noetherian ring and let* G *be a flat affine group scheme over* $\mathbf{k}$*. Let* A *be a finitely generated commutative* $\mathbf{k}$*-algebra on which* G *acts through algebra automorphisms. If* G *is power reductive, then the subring of invariants* A^G *is a finitely generated* $\mathbf{k}$*-algebra.*

Gordan 1868, Hilbert 1890, Hurwitz 1897, Weyl 1926, Emmy Noether 1926, Nagata 1964, Mumford 1965, Serre 1968, Haboush 1975, Springer 1977, Seshadri 1977, Thomason 1987, Franjou–vdK 2010, Jarod Alper 2014.

Nagata's finite generation Theorem

Let $\mathbf{k}$ be a field.

Theorem (Nagata 1964)

Let G be power reductive. If G acts by algebra automorphisms on the finitely generated $\mathbf{k}$ algebra A, then A^G is finitely generated.

Remark

One may reduce to the case that A is graded, generated in degree one, where G respects the grading.

Strategy

Suppose such a graded A is a counterexample. By noetherian induction one may assume that for every nonzero graded G invariant ideal I, the algebra is A/I no counterexample. Power reductivity is then used to derive a contradiction.

Traditional

If $G \to GL_N$ is a homomorphism then it defines a representation of G with underlying $\mathbf{k}$ module $\mathbf{k}^N$.
We call such a representation *traditional*.
On a traditional representation one has coordinates and thus polynomial functions. If $M \twoheadrightarrow \mathbf{k}$ with M traditional, then one can rephrase the question whether $S^n M \twoheadrightarrow S^n \mathbf{k}$ splits in terms of polynomial functions on M. This is what *geometric reductivity* in the 1977 theorem of Seshadri is about.
The 1977 theorem of Seshadri is a theorem and it uses the nice term *geometric reductivity*. However, it distorts the problem posed by Mumford and thus no longer matches the strategy of Nagata.

Resolution

Suppose $N \twoheadrightarrow \mathbf{k}$ is given and one can find a traditional M and a map $M \to N$ so that the composite map $M \to \mathbf{k}$ is surjective.
Then splitting of $S^d M \twoheadrightarrow S^d \mathbf{k}$ implies splitting of $S^d N \twoheadrightarrow S^d \mathbf{k}$.
Thus we encounter the *resolution problem*: If N is finitely generated as a $\mathbf{k}$ module, can we find $M \twoheadrightarrow N$ with M traditional?
[The problem of *equivariant resolution by vector bundles*.]
This reduces an easy problem to a hard one. The hard problem is an interesting problem, but it belongs elsewhere.
The easy problem: Just answer the correct question.
The hard problem: Prove the resolution property [to reduce to the traditional case treated by Seshadri].
If $\mathbf{k}$ is a Dedekind ring then Serre 1968 has established the resolution property.
Thomason 1987 establishes the resolution property in several cases, but under assumptions on the structure of G.

Base change properties

Power reductivity has marvelous base change properties.

Proposition

Let $\mathbf{k} \to S$ *be a map of commutative rings.*

1. *If* G *is power reductive, then so is* G_S.
2. *If* $\mathbf{k} \to S$ *is faithfully flat and* G_S *is power reductive, then so is* G.
3. *If* $G_{\mathbf{k}_\mathfrak{m}}$ *is power reductive for every maximal ideal* $\mathfrak{m}$ *of* $\mathbf{k}$*, then* G *is power reductive.*

Geometric reductivity over a field

Proposition (Popov, Waterhouse)

Let **k** *be a field and let G be an affine algebraic group over* **k**. *The following are equivalent.*

1. *G is power reductive.*
2. *G is geometrically reductive.*
3. *The connected component G^o_{red} of its reduced subgroup G_{red} is reductive.*

Examples

• Recall that a finite flat group scheme G over a commutative ring $\mathbf{k}$ is an affine group scheme over $\mathbf{k}$ whose coordinate ring is a finitely generated projective $\mathbf{k}$-module. They are power reductive.
• Reductive group schemes in the sense of SGA3 are also power reductive. Key case: Chevalley group G over $\mathbb{Z}$.

So the qualitative form ['Hilbert's fourteenth problem'] of the First Fundamental Theorem holds for these examples.

The group scheme GL_N

The *group scheme* GL_N over the base ring $\mathbf{k}$ associates to a commutative $\mathbf{k}$-algebra R the group $GL_N(R)$ of invertible N by N matrices with entries in R.
- Its *coordinate ring* $\mathbf{k}[GL_N]$ is generated over $\mathbf{k}$ by the x_{ij} and $1/\det$, where x_{ij} picks out the ij-th matrix entry. Every $f \in \mathbf{k}[GL_N]$ defines a map $GL_N(R) \to R$, for every R.
- A GL_N-module is a $\mathbf{k}$-module M together with a functorial action of $GL_N(R)$ on $R \otimes_{\mathbf{k}} M$ for all commutative $\mathbf{k}$-algebras R.
- G acts *algebraically*: For every $v \in M$ there are $f_i \in \mathbf{k}[GL_N]$ and $m_i \in M$ such that $g \cdot v = \sum_i f_i(g)m_i$ for $g \in GL_N(\mathbf{k})$, and similarly $g \cdot (1 \otimes v) = \sum_i f_i(g)(1 \otimes m_i)$ for $g \in GL_N(R)$. (Finite sums. This is algebra, not analysis.)

Examples

- $M = \mathbf{k}[GL_N]$ with $(g \cdot f)(x) = f(g^{-1}x)$.
- $\mathbf{k} = \mathbb{Z}$, $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^N, \mathbb{Z}^N \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \wedge^3(\mathbb{Z}^N)$.

Additive group scheme $\mathbb{G}_a$

The additive group scheme $\mathbb{G}_a$ over the base ring **k** represents the functor that associates to a commutative **k**-algebra R its additive group $\mathbb{G}_a(R) := (R, +)$.

Its behaviour is rather different than that of GL_N.

- For instance, when $\mathbf{k} = \mathbb{C}$ we may let $\mathbb{G}_a(\mathbb{C})$ act on the algebra $A = \mathbb{C}[x^2, x^3, y, z]$ such that $t \in \mathbb{C}$ sends x^i to x^i, y to $y + tx^3$, z to $z + tx^2$. The ring of invariants $A^{\mathbb{G}_a}$ has as minimal generating set over $\mathbb{C}$ the set $\{x^\alpha(y - xz)^\beta\}_{\alpha\in\{2,3\},\beta\geq 0}\}$. So $A^{\mathbb{G}_a}$ is not finitely generated (Neena Gupta). ✗ (Modern).

Maurer 1899: $\mathbb{G}_a \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$ ✓(Traditional)

Nagata counterexample 1958: ${\mathbb{G}_a}^{13} \curvearrowright \mathbb{C}[x_1, \ldots, x_{32}]$ ✗ (Traditional)

Totaro 2008: ${\mathbb{G}_a}^3 \curvearrowright F[x_1, \ldots, x_{18}]$ ✗ (Traditional)

- The only irreducible representation of $\mathbb{G}_a$ over $\mathbb{C}$ is the trivial representation $\mathbb{C}$, so most representations of $\mathbb{G}_a$ are not completely reducible.

The End

THANK YOU!