A Friedlander-Suslin theorem over a noetherian base ring

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Theorems of Evens and of Friedlander-Suslin.

• Let k be a commutative noetherian ring. Let G be a finite group over k acting on a finitely generated commutative k-algebra A. The (CFG) Theorem of Leonard Evens (TAMS 1961) says

Theorem

$H^*(G, A)$ is a finitely generated k-algebra.

• Now let k be a field. Let G be a finite group scheme over k acting on a finitely generated commutative k-algebra A. Theorem of Eric Friedlander and Andrei Suslin (Invent. Math. 1997).

Theorem

$H^*(G, A)$ is a finitely generated k-algebra.

To show: The theorem is still valid if k is a commutative noetherian ring arXiv:2212.14600. But *finite group scheme over k* = affine group scheme that is finite *and flat* as a scheme over k.

My conjecture. Franjou and Touzé.

Let k be a field again. Let G be a reductive group scheme over k. As always let G be acting on a finitely generated commutative k-algebra A. My conjecture (2004) says

Conjecture

$H^*(G, A)$ *is a finitely generated* k*-algebra.*

Reason: I had believed the opposite, but mathematics was not cooperating.
The conjecture was proven by Antoine Touzé (Duke Math J. 2010). Antoine Touzé was a student of Vincent Franjou. That is how I got to work with Franjou.

Work with Srinivas.

The proof of my conjecture had several parts. The work of Touzé provided the last and most difficult piece of the puzzle.
One of the earlier ingredients was work with Srinivas using functorial resolutions of the sheaf of the diagonal in a product of Grassmannians.
Let G be the group scheme GL_N over the field k. Recall that V has *good filtration* if and only if $H^{>0}(SL_N, V \otimes k[SL_N/U])$ vanishes.
Now our theorem says that if A is a finitely generated commutative k-algebra with good filtration and M is a finitely generated G-equivariant A-module, then M has a finite resolution by modules with good filtration. In particular $H^m(G, M)$ vanishes for large m.

Significant overlap.

Suppose that instead of a field k we take a commutative noetherian ring k.
Changing the definition of the term *negligible*, the proof from the joint paper with Srinivas goes through with only very minor modifications. The arXiv issued an "arXiv admin note" about *significant overlap* of the TEX files.
[Good Grosshans filtration in a family, In: Autour des schémas en groupes (Group Schemes, a celebration of SGA3), Volume III. Panoramas et Synthèses 47 (2015), 111-129.] with 38 occurrences of *negligible*.

Power reductivity.

With Franjou I worked on $A^G = H^0(G, A)$, in particular for $G = GL_N$ over a noetherian ring k. We found that *power reductivity* is the right notion to get finite generation of invariants. (G is called power reductive if, whenever $A \twoheadrightarrow B$ is a surjective map of commutative k-algebras with G action, and $b \in B^G$, then some power of b comes from A^G.)
No work of Bob Thomason on resolution properties is needed.
"Avoid taking duals of representations! Return from Seshadri to Mumford."

Hints by Benson on torsion.

Then we turned to $H^*(G, A)$ with k noetherian, $G = GL_N$, hoping to profit from the work of Touzé. David Benson explained that in his 1987 work with Nathan Habegger there are tricks about torsion in cohomology of groups.

Suppose there is a positive integer n that annihilates all of $H^{>0}(G, A)$. If p is a prime number, then we learn that if $m > 0$, then every $\alpha \in H^m(G, A/pA)$ has a power that comes from $H^{>0}(G, A)$.

This gives a link between the algebras $H^*(G, A)$ and $H^*(G, A/pA)$. But k/pk contains a field and thus we are closer to the work of Touzé.

Bounded torsion gives finite generation.

Combining with my "negligible" work we get (2015):

Theorem

Let A be a finitely generated commutative k-algebra with GL_N action over the noetherian ring k.
If there is a positive integer n that annihilates $H^{>0}(GL_N, A)$, then $H^*(GL_N, A)$ is a finitely generated k-algebra.

If there is such n, then we say that $H^{>0}(GL_N, A)$ has *bounded torsion*.

Induce up to GL_N.

Return to the situation where k is noetherian and G is a finite flat group scheme. Our aim is to show that the statement of the Friedlander-Suslin theorem still holds. Embed G into some GL_N. We know that GL_N/G is affine and that

$$H^{>0}(G, A) = H^{>0}(GL_N, \mathrm{ind}_G^{GL_N} A).$$

So let us show that the left hand side has bounded torsion.
We should have done that years ago? Srinivas visited recently and asked questions.
Note that bounded torsion is obvious when k contains $\mathbb{Z}/n\mathbb{Z}$ for some $n > 0$. So we further assume k contains $\mathbb{Z}$.

Averaging applied to an invariant vector.

How is it done when G is a constant group scheme of order n? A down to earth way to tell it uses the element $\sum_{g\in G} g$ of the group ring kG. The group ring is dual to the coordinate ring $k[G]$ of the group scheme. Say $V \twoheadrightarrow W$ is a surjective map of G-modules and $w \in W^G$. Pick $v \in V$ with image w. Then $\sum_{g\in G} gv \in V^G$ maps to nw. We see that n annihilates the cokernel of $V^G \to W^G$. It follows by dimension shift that

Theorem

n annihilates $H^{>0}(G, M)$ for every G-module M.

We need to generalise this proof to the case that G is a finite flat group scheme over a noetherian ring k containing $\mathbb{Z}$.

Use Hopf algebra facts.

So we need to check the literature on Hopf algebras. We find a suitable 1971 paper by Bodo Pareigis. Bodo Pareigis reviewed my 1971 paper in the Zentralblatt. The circle is full.
Anyway, Pareigis shows that the “left integrals” in kG span a summand that is a projective k-module of rank one. A left integral $\psi \in kG$ defines a G-module map $k[G] \to k$. Equivalently, it is a $\psi \in kG$ with $\chi\psi = \chi(1)\psi$ for $\chi \in kG$.
Think of the left invariant measure dg in a Reynolds operator $v \mapsto \int_G gv\, dg$. But we now want the “volume” $\psi(1)$ of G to be a positive integer.
Our bounded torsion problem is local on $\mathrm{Spec}(k)$, so we may assume the projective module is free, say with generator ψ. We must show that some multiple of ψ sends $1 \in k[G]$ to a positive integer n. We may tensor with $\mathbb{Q}$ and investigate at geometric points. There we are back at group rings (Cartier's theorem). □

Not a conjecture.

Now let $k = \mathbb{Z}$ or some DVR of unequal characteristic. Let $N > 2$. Let GL_N act on a finitely generated commutative k-algebra A.

Problem

Is $H^*(GL_N, A)$ always a finitely generated k-algebra?
Equivalently, does $H^{>0}(GL_N, A)$ always have bounded torsion?

We have no guess.

The End

THANK YOU!