Two triangular matrices

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The two matrices

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$$\mathcal{P}_v = \mathcal{L}(P(-e_v)) \text{ and } \mathcal{Q}_v = \mathcal{L}(Q(e_v)).$$

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We are interested in $\langle[\mathcal{P}_v],[\mathcal{Q}_w]\rangle$, for semi-orthogonality.
Say $\mathrm{k} = \mathbb{C}$. Write $\alpha_{vw} = \langle[\mathcal{O}_{X_w^+}],[\mathcal{Q}_v]\rangle$. So

$$[\mathcal{Q}_v] = \sum \alpha_{vw}[\mathcal{O}_{X^w}(-\partial X^w)]$$

Write $\beta_{vw} = \langle[\mathcal{O}_{X^w}(-\partial X^w)],[\mathcal{P}_v]\rangle$. So

$$[\mathcal{P}_v] = \sum \beta_{vw}[\mathcal{O}_{X_w^+}]$$

Our main result concerning these auxiliary matrices is that, with a suitable reordering of rows and columns, the matrices (α_{wv}) and (β_{vw}) are upper triangular and invertible. And that we know enough entries to compute the needed $\langle[\mathcal{P}_v],[\mathcal{Q}_w]\rangle$.

Rewriting

Let us start with (β_{vw}). Write $\langle[\mathcal{O}_C],[\mathcal{F}]\rangle$ as $\chi(C,\mathcal{F})$ when C is a T-invariant closed subset of $\mathcal{B}=G/B$ and $\mathcal{F}$ is a T-equivariant coherent sheaf.

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So $(\beta_{vw})=\chi(X^w,\mathcal{L}(P(-e_v)))-\chi(\partial X^w,\mathcal{L}(P(-e_v)))=$ $\chi(\overline{Bww_0B}/B,\mathcal{L}(\Gamma(X_y,\mathcal{L}_\lambda)))-\chi(\partial\overline{Bww_0B}/B,\mathcal{L}(\Gamma(X_y,\mathcal{L}_\lambda)))$, with λ dominant, $y\lambda=-e_v$.

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We get to study $\chi(S_Y/B,\mathcal{L}(\Gamma(X_z,\mathcal{L}_\lambda)))$ when λ is dominant, $z\in W$ and $S_Y:=\bigcup_{w\in Y}\overline{BwB}$ for some subset Y of W.
We claim the *contraction property*
$\chi(S_Y/B,\mathcal{L}(\Gamma(X_z,\mathcal{L}_\lambda)))=\chi(S_Y\overline{BzB}/B,\mathcal{L}_\lambda)]$.

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By Ramanathan $\mathcal{O}_{G/B}\overset{\curvearrowleft}{\to}F_*\mathcal{O}_{G/B}$ the intersections $(S_Y/B)\cap(S_V/B)$ are reduced. [Invent. Math. 1985]

Therefore we have the Mayer-Vietoris relation
$[\mathcal{O}_{S_Y/B}]+[\mathcal{O}_{S_V/B}]=[\mathcal{O}_{S_Y\cup S_V/B}]+[\mathcal{O}_{S_Y\cap S_V/B}]$.

Proof of contraction property

The contraction property is well-known when Y is a singleton ([Polo 1989, Prop. 1.4.2], Demazure operators, Bott-Samelson-Demazure-Hansen resolution, ...). So we can argue by induction along the poset of the S_Y provided that both sides in the contraction claim satisfy a Mayer-Vietoris law.

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We need the *distributive laws*

$(S_Y \cup S_V)\overline{BzB} = (S_Y\overline{BzB}) \cup (S_V\overline{BzB})$.

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So let $\overline{BuB} \subset (S_Y\overline{BsB}) \cap (S_V\overline{BsB})$. Then $u \leq y \star s$ for some $y \in Y$ and $u \leq x \star s$ for some $x \in V$.

Let u' be the unique minimal coset representatative of $u < s >$.

Then $\overline{BuB} \subset \overline{BuB}\ \overline{BsB} = \overline{Bu'B}\ \overline{BsB}$ with $\overline{Bu'B} \subset (S_Y \cap S_V)$.

Rappels: Minimal coset representatives

Let λ be a dominant weight. Let I be the set of simple roots perpendicular to λ. Then the stabilizer in W of λ is the subgroup W_I generated by the s_α with $\alpha \in I$. One calls W_I a parabolic subgroup of W.

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Every coset wW_I has a unique shortest element, known as the minimal coset representative. An element w is the minimal coset representative of wW_I if and only if $w\alpha > 0$ for $\alpha \in I$.
One denotes by W^I the set of minimal coset representatives.
$(u, v) \mapsto uv$ is a bijection $W^I \times W_I \to W$, with $\ell(uv) = \ell(u) + \ell(v)$, and $W \to W^I$ respects Bruhat order.

Extremal weights

Thanks to the contraction property we may view (β_{vw}) as
$\chi(\overline{Bww_0B}X_y, \mathcal{L}_\lambda))) - \chi(\partial\overline{Bww_0B}X_y, \mathcal{L}_\lambda)))$,
with λ dominant, $y\lambda = -e_v$.

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Ramanan–Ramanathan tells that the $H^i(S_V/B, \mathcal{L}_\lambda)$ vanish for $i > 0$ and that $\Gamma(G/B, \mathcal{L}_\lambda) \to \Gamma(S_V/B, \mathcal{L}_\lambda)$ is surjective.
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We also know that $\Gamma(S_V/B, \mathcal{L}_\lambda)$ has a filtration by $Q(\mu)$ with μ running over the extremal weights of $\Gamma(S_V/B, \mathcal{L}_\lambda)$.
The extremal weights that occur have multiplicity one.
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To determine β_{vw} it suffices to know the characters of the $Q(\mu)$ that occur in $\Gamma(\overline{Bww_0B}X_y, \mathcal{L}_\lambda)$ but not in $\Gamma(\partial\overline{Bww_0B}X_y, \mathcal{L}_\lambda)$.

Zeroes in β matrix on one side of the 'diagonal'

So to get β_{vw}, we need to find the complement of $\{x\lambda \mid x \leq z \star y \text{ for some } z < ww_0\}$ in $\{x\lambda \mid x \leq (ww_0) \star y\}$. Here $\lambda = -w_0 v e_v$, $y = v^{-1} w_0$.

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Say $vw_0 \not\le w$. Then $\ell((ww_0) \star y)$ is not $\ell(ww_0) + \ell(y)$, because otherwise there would be $u \in W$ with $\ell(u) + \ell(ww_0) + \ell(y) = \ell(uww_0 y) = \ell(w_0)$, so $u \star (ww_0) = w_0 y^{-1} = v$, so $v \ge ww_0$, $vw_0 \le w$.

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As $\ell((ww_0) \star y)$ is not $\ell(ww_0) + \ell(y)$, there is a $z < ww_0$ for which $z \star y$ and $(ww_0) \star y$ are equal. Therefore β_{vw} vanishes if $vw_0 \not\leq w$ and the β matrix is triangular after rearranging rows and columns: $\beta_{vw_0,w} = 0$ if $vw_0 \succ w$.

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On the 'diagonal' we expect invertible elements, because we are comparing two bases of $K_T(G/B)$. (That the classes of the $[\mathcal{P}_v]$ generate $K_T(G/B)$ is known from a different story.)

When $vw_0 = w$

Let $vw_0 = w$, $\lambda = -w_0 v e_v$, $y = v^{-1}w_0$. Again we need to find the complement of $\{x\lambda \mid x \leq z \star y \text{ for some } z < ww_0\}$ in $\{x\lambda \mid x \leq ww_0 \star y\}$.
We claim that this complement is the singleton $\{-ve_v\}$.

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Now suppose $-ve_v$ is in $\{x\lambda \mid x \leq z \star y \text{ for some } z < v\}$. We may always take z smaller so that $z \star y = zy$.

Then $-ve_v = x(-w_0 ve_v)$, so xw_0 lies in the parabolic subgroup W_I of elements that fix ve_v. And $xw_0 \geq zyw_0 = zv^{-1}$. But v^{-1} is a minimal coset representative and z^{-1} would be smaller. This is absurd. So the complement contains $-ve_v$.

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There cannot be more in the complement, as that would spoil the invertibility of β_{v,vw_0} in $R(T)$.

Sanity check

But let us look anyway at $-uve_v$ with $-uve_v \neq -ve_v$. We may take u minimal. As $\ell(u) \geq 1$, there is a simple refection $s = s_\alpha$ with $\ell(us) = \ell(u) - 1$ and $sve_v \neq ve_v$. From the definition of e_v we see that $v^{-1}\alpha < 0$, so $v^{-1}s < v^{-1}$. Put $z = sv$, $x = uw_0$. Then $z < v = ww_0$, $z \star y = (sv) \star v^{-1}w_0 = sw_0$, $-uve_v = x\lambda$, $x = uw_0 \leq sw_0 = z \star y$.
So $-uve_v$ lies in $\{x\lambda \mid x \leq z \star y \text{ for some } z < ww_0\}$.

A 'diagonal' element

We are still looking at β_{v,vw_0}.
If $vw_0 = w$, we have found that only the character of $Q(-ve_v)$ remains, but we expected an invertible element, because we are comparing two bases of $K_T(G/B)$.
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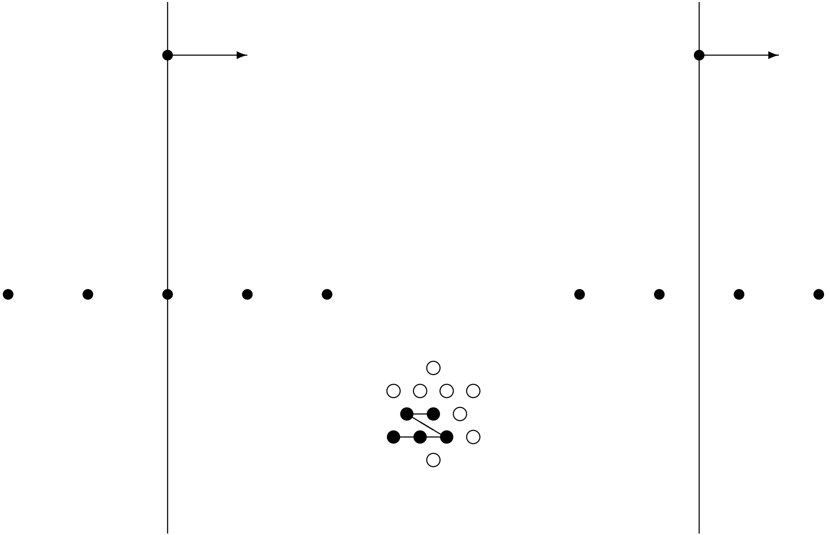
Indeed $Q(-ve_v)$ is one dimensional.
Proof: We may assume to be working over $\mathbb{C}$. Now $Q(-ve_v)$ has B-socle of weight $-ve_v$ and all other weights are strictly smaller. If there is another weight, then there also must be another weight of the form $-ve_v + n\alpha$ with α simple. That is because we can get from any weight of $Q(-ve_v)$ to $-ve_v$ by acting with elements $X_{-\beta} \in \mathfrak{b}$, where β is a simple root. But $-ve_v$ is too close to the s_α reflection hyperplane. See picture.
This ends the discussion of the β matrix.

The reflection hyperplane



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Now at x_+ the fiber of $\mathcal{Q}_v$ is $Q^+(w_0 e_v)$, so
$\mathcal{Q}_v = \ker : \mathcal{L}^+(\Gamma(X_z^+, \mathcal{L}_\lambda^+)) \to \mathcal{L}^+(\Gamma(\partial X_z^+, \mathcal{L}_\lambda^+))$ with λ *anti-dominant* and $z\lambda = w_0 e_v$.

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So we want to know the difference between
$\langle[\mathcal{O}_{X_w^+}], [\mathcal{L}^+(\Gamma(X_z^+, \mathcal{L}_\lambda^+))]\rangle$ and $\langle[\mathcal{O}_{X_w^+}], [\mathcal{L}^+(\Gamma(\partial X_z^+, \mathcal{L}_\lambda^+))]\rangle$

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when $S_Y^+ := \bigcup_{w \in Y} \overline{B^+ w B^+}$ for some subset Y of W.
We claim the *contraction property*
$\langle [\mathcal{O}_{X_z^+}], [\mathcal{L}^+(\Gamma(S_Y^+/B^+, \mathcal{L}_\lambda^+))] \rangle = \langle [\mathcal{O}_{\overline{B^+ z B^+ S_Y^+}/B^+}, \mathcal{L}_\lambda^+))] \rangle$

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$\overline{B^+zB^+}(S_Y^+ \cap S_V^+) = (\overline{B^+zB^+}S_Y^+) \cap (\overline{B^+zB^+}S_V^+)$.

The non-obvious inclusion is

$\overline{B^+zB^+}(S_Y^+ \cap S_V^+) \supset (\overline{B^+zB^+}S_Y^+) \cap (\overline{B^+zB^+}S_V^+)$.

Apply the map $g \mapsto g^{-1}$.

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Here $\lambda = w_0 v e_v$, $z = w_0 v^{-1} w_0$.
If $w \not\leq v w_0$, then $\ell(w \star z)$ is not $\ell(w) + \ell(z)$, because otherwise there would be $u \in W$ with $u \star w \star z = uwz = w_0$, $\ell(u) + \ell(w) + \ell(z) = \ell(w_0)$, $uw = vw_0$, $w \leq vw_0$.

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When $\ell(w \star z)$ is not $\ell(w) + \ell(z)$, there is a $y < z$ so that $w \star y$ and $w \star z$ are equal. Therefore α_{vw} vanishes if $w \not\leq vw_0$ and the α matrix is triangular after rearranging rows and columns.

When $w = vw_0$

Now let $w = vw_0$. We claim that the complement of $\{x\lambda \mid x \leq w \star y \text{ for some } y < z\}$ in $\{x\lambda \mid x \leq w \star z\}$ is the singleton $\{ve_v\}$.

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Now suppose there are $y < z$ and $x \leq w \star y$ with $x\lambda = ve_v$. Replacing y by a lesser element we may assume $w \star y = wy = vw_0 y$. Then xw_0 fixes ve_v, $x \leq vw_0 y$, so $xw_0 \geq vw_0 y w_0$. As xw_0 lies in the parabolic subgroup W_I of elements fixing ve_v, we must have $vw_0 y w_0 \in W_I$. And $y < z$, so $w_0 y w_0 < w_0 z w_0 = v^{-1}$. Thus $w_0 y w_0$ is shorter than the minimal coset representative v^{-1}. This is absurd.

So the complement contains at least ve_v. Again there cannot be more, as α_{v,vw_0} must be invertible.

Sanity check

But let us consider a u with $u \leq w \star z$ and $uve_v \neq ve_v$. We want to show that uve_v is in the subset $\{x\lambda \mid x \leq w \star y \text{ for some } y < z\}$.
We may replace u with the minimal coset representative in its coset of the stabilizer W_I of ve_v. As $\ell(u) \geq 1$ there is a simple refection $s = s_\alpha$ with $\ell(us) = \ell(u) - 1$ and $sve_v \neq ve_v$. From the definition of e_v we see that $v^{-1}\alpha < 0$, so $v^{-1}s < v^{-1}$.
Recall that $\lambda = w_0 ve_v$, $z = w_0 v^{-1} w_0$, $w = vw_0$.
Put $y = w_0 v^{-1} sw_0$, $x = uw_0$. Then $y < z$,
$w_0(w\star y)w_0 = w_0((vw_0)\star(w_0 v^{-1} sw_0))w_0 = (w_0 v)\star(v^{-1}s) = w_0 s$.
And $x \leq sw_0 = w \star y$, $x\lambda = uve_v$. Done.

The End

THANK YOU!