Invariants

P. Gordan 1868,
J.f.d. reine u. angew. Math., 69
Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.

In modern language:
$G = SL_2(\mathbb{C})$ as algebraic group.
$G \curvearrowright V := \mathbb{C}^2, \quad \mathbb{C}[V] = \mathbb{C}[X,Y], \quad W_d := \mathbb{C}[V]_d,$
$W_2 = \{aX^2 + bXY + cY^2\}, \quad \mathbb{C}[W_2] = \mathbb{C}[a,b,c],$
$b^2 - 4ac \in \mathbb{C}[W_2]^G$ an *invariant* (= fixed point).

$\mathbb{C}[W_d]^G$ is finitely generated (f.g.) as a $\mathbb{C}$-algebra and $(W_d \otimes \mathbb{C}[W_d])^G$ is a noetherian $\mathbb{C}[W_d]^G$-module.

Nowadays

we know more generally:
G reductive linear algebraic group over $\mathbb{C}$.
A f.g. $\mathbb{C}$-algebra, $G \curvearrowright A$, then A^G is f.g.
Consequently, if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module.

Traditional case: $A = \mathbb{C}[V]$, $\quad M = W \otimes A$, $G \curvearrowright V, W$ linear, $\dim V < \infty$, $\dim W < \infty$. Then M^G is noetherian over the f.g. A^G. Observe that now the algebra A has no zero divisors and that M is flat over A. Note that $M \subseteq \mathbb{C}[V \oplus W^{\#}]$.

The traditional case is different: Finite generation of invariants even holds when G is the additive subgroup $\mathbb{G}_a$ of $SL_2(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ (Weitzenböck 1932 at Amsterdam). By Jacobson-Morozov the action then extends to $SL_2(\mathbb{C})$ and a result of Grosshans 1983 applies.

How we got there

Hilbert 1890 Math. Annalen 36
Finite generation (the traditional case) for $G = SL_n(\mathbb{C})$, using the Ω process of Cayley and noetherian arguments. Nonconstructive.

Hurwitz 1897 considers compact group K with Haar measure dk and replaces the Ω process with the method of averageing. $K \curvearrowright V$ linear. Get linear equivariant retract $V \to V^K$ from

$$v \mapsto \frac{\int_K kv\, dk}{\int_K dk}.$$

Weyl 1926 takes in a semi-simple complex Lie group G a maximal compact subgroup K and notes that the retract is also G-equivariant from V to V^G.

One can handle any base field of characteristic zero. (faithfully flat base change).

Characteristic $p > 0$

Let k be a field of characteristic p.
E. Noether 1926 considers a finite group G.
If A is f.g. k-algebra, $G \curvearrowright A$, then A^G is f.g.
Also if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module.
Further A is integral over A^G.

Mumford GIT 1965. Say reductive G acts on the affine variety Spec(A). One wants to form the quotient Spec(A)$/G$, hopes it is affine and in fact equal to Spec(A^G). In particular one wants again that A^G is f.g. But equivariant linear retracts are no longer available.
Consider $G \curvearrowright V$ linear, $\dim V < \infty$, $L \subseteq V^G$ one dimensional. Then Mumford asks for an equivariant map (polynomial) $f : V \to L$, so that $f|L$ is non-constant.
If such f always exists, call G *geometrically reductive*.

Necessity

Consider $k[L]$ as $k[V]$-module through restriction $\phi \mapsto \phi|L$. Then
f exists iff $k[L]^G$ is noetherian module for $k[V]^G$.

Nagata 1964 showed that if G is geometrically reductive, the finite generation theorems hold and, if G is a subgroup of some group L, he argued that $L/G = \text{Spec}(k[L]^G)$ is affine.

Haboush 1975
'Reductive groups are geometrically reductive.'

Borsari & Ferrer Santos 1992 observed that the f.g. proofs of Nagata work for any geometrically reductive affine group scheme $G \subseteq GL_n$.

In particular, if G is a finite group scheme, i.e. $\dim(k[G]) < \infty$, then this applies. One can show directly that *finite group schemes are geometrically reductive*. For finite generation one can more easily follow Noether and show that A is integral over A^G.

Cohomology.

Evens 1961 takes a finite group G and generalizes the result of Noether: If A is f.g. k-algebra, $G \curvearrowright A$, then $H^*(G,A)$ is f.g.
Also if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then $H^*(G,M)$ is noetherian $H^*(G,A)$-module.
This result is important as starting point of the theory of *support varieties*.

For modular representation theory of algebraic groups *Frobenius kernels* are important.
For example for $G = SL_n$ one puts (scheme theoretically) $G_r = \ker F^r : G \to G$, so

$$G_r := \{(\, a_{ij}\,) \mid (\, a_{ij}^{p^r}\,) = 1_n\} \subset SL_n\,.$$

A G_1-module is the same as a module for the Lie algebra $\mathfrak{g}$ with its $[p]$-structure. Unlike in characteristic zero the first order is not enough and one needs the higher G_r also.

$$\varprojlim \mathrm{Ext}^*_{G_r}(V,W) = \mathrm{Ext}^*_G(V,W)$$

if V, W are finite dimensional. (Friedlander-Parshall 1987,... [Jantzen 1987])

For a theory of support varieties of semisimple groups one needs a proof for the generalization of Evens 1961 to finite group schemes. This *"proved surprisingly elusive"*.

Friedlander–Suslin 1997. Let G be a finite group scheme. The following theorem is curiously missing, for nonconnected G. (What they do state suffices as starting point for the theory of support varieties.)
If A is f.g. k-algebra, $G \curvearrowright A$, then $H^*(G, A)$ is f.g.
Also if M is noetherian A-module, $G \curvearrowright M$,
$A \otimes M \to M$ equivariant, then
$H^*(G, M)$ is noetherian $H^*(G, A)$-module.

Let us say that an affine group scheme has property CFG (cohomological finite generation) if $H^*(G, A)$ is f.g. when A is f.g. k-algebra, $G \curvearrowright A$.

So finite group schemes have CFG by Friedlander–Suslin, "missing statement".

Lemma

If the linear algebraic group G over k satisfies CFG and H is a geometrically reductive subgroup scheme, then H satisfies CFG.

Proof. $H^*(H,A) = H^*(G, \mathrm{ind}_H^G(A))$ because G/H is affine. Now $\mathrm{ind}_H^G(A) = (A \otimes k[G])^H$ is f.g.

Conjecture

SL_n satisfies CFG.

Corollary to conjecture

Any geometrically reductive affine group scheme satisfies CFG. (and conversely).

G, B, T, U.

Let G be semisimple, B a Borel subgroup, T a maximal torus in B, U the unipotent radical of B. Because the conjecture is about SL_n, feel free to take $G = SL_n$, B the subgroup of upper triangular matrices, T the subgroup of diagonal matrices, U the subgroup of upper triangular matrices with ones on the diagonal.

We put an additive height function on the weight lattice $X(T) = \mathrm{Hom}(T, GL_1)$

$$\mathrm{ht} : X(T) \to \mathbb{Z},$$

say $\mathrm{ht} = 2\sum_{\alpha>0}\alpha^\vee$. The height of positive roots is strictly positive.

If V is a G-module, possibly of infinite dimension, let $V_{\leq i}$ denote the largest G-submodule all whose weights have height $\leq i$. Thus $V_{\leq 0} = V^G$.

The associated graded module we call the *Grosshans graded* of V, (invented earlier by Mathieu or Popov)

$$\mathrm{gr}\, V = \oplus_{i\geq 0} V_{\leq i}/V_{\leq i-1}.$$

If A is f.g. and $G \curvearrowright A$, then Grosshans 1992 shows that A^U and $\mathrm{gr}\, A$ are f.g.

For the finite generation of A^U one uses the transfer principle:
$A^U = (\mathrm{ind}_U^G A)^G = (A \otimes k[G/U])^G$ where the *multicone* $k[G/U]$ equals

$$\oplus_{[\mathcal{L}]\in\mathrm{Pic}}\Gamma(G/B,\mathcal{L}) = \oplus_{\lambda\in X(T)}\nabla(\lambda).$$

If λ is dominant then $\nabla(\lambda)$ is the *costandard module* of highest weight λ. It is the largest G-module M with $\dim M^U = 1$ and highest weight λ. (Weyl character formula gives its character.)

We say that V has *good filtration* if $\mathrm{gr}\, V$ is a direct sum of costandard modules. Unlike Friedlander 1985 we do not require $\dim V \leq \aleph_0$.

Modules with good fitration form an important class, with some remarkable properties.
If V, W have good filtration, so does $V \otimes W$, by Wang Jian-Pan 1982, Donkin 1985, Mathieu 1990.

After Friedlander–Parshall 1986 we say that V has *good filtration dimension at most* d, we write $\dim_\nabla(V) \leq d$, if there is a resolution

$$V \to M_0 \to \cdots \to M_d \to 0$$

where the M_i have good filtration.

Cohomological criterion

$$\dim_\nabla(V) \leq d \Leftrightarrow H^{d+1}(G, V \otimes k[G/U]) = 0.$$

Note that $H^i(G, V) = 0$ for $i > \dim_\nabla(V)$.

Connection between CFG and $\dim_\nabla$

Now suppose G satisfies CFG, A is f.g., $G \curvearrowright A$ with good filtration, M is noetherian A-module with compatible G action. We claim that then $\dim_\nabla(M) < \infty$.

Indeed $H^*(G, A \otimes k[G/U])$ is just $(A \otimes k[G/U])^G$ and $H^*(G, M \otimes k[G/U])$ is by CFG a noetherian module over it. In particular, $H^*(G, M \otimes k[G/U])$ lives in only finitely many degrees and we find that $\dim_\nabla(M) < \infty$.

The conclusion

$$\dim_\nabla(A) = 0 \Rightarrow \dim_\nabla(M) < \infty$$

surprised me. It looked like negative evidence for the conjecture. Let us try to see why $\dim_\nabla(M)$ would be finite. We do not see how to embed M into any *noetherian* A-module with good filtration. So we turn around and try a projective resolution. Indeed if W is a finite dimensional G-submodule that generates M as an A-module, we have a surjection $W \otimes A \to M$. Repeating, we get a projective resolution of M. But this almost never stops, as M rarely has finite projective dimension.

Thus the conclusion $\dim_\nabla(M) < \infty$ did not look plausible before the 60th birthday of Eric. I still do not understand. Nevertheless I proved it after the birthday under the

Key hypothesis

The symmetric algebra $S^*(\nabla(\varpi_i))$ has good filtration for every fundamental weight ϖ_i.

The key hypothesis is satisfied when $G = SL_n$, $n \leq 5$ and also when $p \geq \dim(\nabla(\varpi_i))$ for all ϖ_i. For instance, if G is of type E_8, then $p = 6899079289 > 6899079264$ is sufficiently large.

Let us mention some ingredients of the proof. One turns to $\operatorname{gr} M$ and $\operatorname{gr} A$. Using the key hypothesis one reduces to a situation where the module is a module over a polynomial ring. Then projective resolutions do stop and one uses that projectives are free.

The trick is to find a diagonalizable group scheme D and make $G \times D$ act on a tensor product

$$P = \bigotimes S^*(\nabla(\varpi_i))^{\otimes m_i}$$

in such a manner that there is a surjection $P^D \to \operatorname{gr} A$. View $\operatorname{gr} M$ as the summand of D-invariants in $P \otimes_{P^D} \operatorname{gr} M$. It remains to deal with the module $P \otimes_{P^D} \operatorname{gr} M$ over the polynomial ring P.

What about the conjecture itself? We can say more about the flat deformation $\operatorname{gr} A$ of A than about A itself. Say $G = SL_n$. Under the key hypothesis we can show that at least

$H^*(G, \operatorname{gr} A)$ is finitely generated,

for our f.g. A.

The starting point is a result of Mathieu which says that $\operatorname{gr} A$ is a noetherian module over a Frobenius twist of an algebra with good filtration. The Frobenius twist makes that we get to look at the precise results of Friedlander–Suslin. We then show the good filtration dimension of $H^*(G_r, \operatorname{gr} A)$ is finite and apply the Hochschild–Serre spectral sequence.

Only for $G = SL_2$ or for $p = 2$, $\quad G = SL_3$ can we pass to A itself and prove the conjecture. This 'degrading' is achieved by means of a set of universal cohomology classes $c[m]$ that interpolates the set constructed by Friedlander–Suslin.

Recall that the divided power $\Gamma^m(V)$ consists of the subspace of symmetric tensors in $V^{\otimes m}$. In general we would like to have classes $c[m] \in H^{2m}(GL_n, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ that lift the m-th cup power $c[1]^{\cup m} \in H^{2m}(GL_n, (\mathfrak{gl}_n^{(1)})^{\otimes m})$ of a certain class $c[1]$.

This Witt vector class $c[1]$ classifies the extension

$$1 \to \mathfrak{gl}_n^{(1)} \to GL_n(W_2(k)) \to GL_n(k) \to 1,$$

where $W_2(K)$ is the ring of length two Witt vectors.

But this may be entirely the wrong way. An alternative would be to try and show that $H^*(SL_n, A) \to H^*(SL_n, A/J)$ is surjective up to repeated pth powers, when J is an invariant ideal.
Anyway, a better understanding would help. Eric?

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