Power Reductivity, Heeze 2009

with Vincent Franjou

P. Gordan 1868, J.f.d. reine u. angew. Math., 69, 323–354.
Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.

In modern language: $G = SL_2(\mathbb{C})$ as algebraic group.
$G \curvearrowright V := \mathbb{C}^2, \quad \mathbb{C}[V] = \mathbb{C}[X, Y],$
$W_d := \mathbb{C}[V]_d, \; W_2 = \{aX^2 + bXY + cY^2\}, \quad \mathbb{C}[W_2] = \mathbb{C}[a, b, c],$
$b^2 - 4ac \in \mathbb{C}[W_2]^G$ an *invariant* (= fixed point).

$\mathbb{C}[W_d]^G$ is finitely generated (f.g.) as a $\mathbb{C}$-algebra and
$(W_d \otimes \mathbb{C}[W_d])^G$ is a noetherian $\mathbb{C}[W_d]^G$-module.

Nowadays

we know more generally:

G reductive linear algebraic group over $\mathbb{C}$. $G \curvearrowright A$,
A f.g. $\mathbb{C}$-algebra, then A^G is f.g.
(Actions are *rational*: $A \to A \otimes \mathbb{C}[G]$)
Consequently, if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module.
(use algebra $A \ltimes M$)

Traditional 'linear' case:
$A = \mathbb{C}[V]$, $M = W \otimes A$, $G \curvearrowright V, W$ linear, $\dim V < \infty$, $\dim W < \infty$.
Then M^G is noetherian over the f.g. A^G. Observe that now the algebra A has no zero divisors and that M is flat over A.

The traditional case is different:
Finite generation of invariants even holds when G is the algebraic additive group $\mathbb{G}_a$.
(Weitzenböck 1932 at Amsterdam = Maurer 1899).
You can think of G as subgroup of $SL_2(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. An algebraic action of $\mathbb{G}_a$ on V is given by a nilpotent linear map $V \to V$. By Jacobson-Morozov the action then extends to $SL_2(\mathbb{C})$ and a result of Grosshans 1983 applies.

How we got there

Hilbert 1890 Math. Annalen 36
Finite generation (the traditional case) for $G = SL_n(\mathbb{C})$, using the Ω process of Cayley and noetherian arguments. Nonconstructive.

Hurwitz 1897 considers compact group K with Haar measure dk and replaces the Ω process with the method of averaging. $K \curvearrowright V$ linear. Get linear equivariant retract $V \to V^K$ from

$$v \mapsto \frac{\int_K kv\ dk}{\int_K dk}.$$

Weyl 1926 takes inside any semi-simple complex Lie group G a maximal compact subgroup K and notes that the retract of Hurwitz is also G-equivariant from V to V^G.

One can handle any base field of characteristic zero. (faithfully flat base change).

E. Noether 1926 considers a finite group G and basically works over a commutative noetherian ring k.
If A is f.g. k-algebra, $G \curvearrowright A$, then A^G is f.g.
Further A is integral over A^G.

Also if M is noetherian A-module, $G \curvearrowright M$, $A \otimes M \to M$ equivariant, then M^G is noetherian A^G-module.

Characteristic $p > 0$

Let k be a field of characteristic p.

Mumford GIT 1965. Say reductive G acts on the affine variety Spec(A). One wants to form the quotient Spec(A)/G, hopes it is affine and in fact equal to Spec(A^G). In particular one wants again that A^G is f.g. But equivariant linear retracts are no longer available.

Mumford Conjecture. Consider $G \curvearrowright V$ linear, $\dim V < \infty$, $L \subseteq V^G$ one dimensional. Then Mumford asks for an equivariant map (polynomial) $f : V \to L$, so that $f|L$ is non-constant.
If such f always exists, call G *geometrically reductive*.
The conjecture is then that reductive groups like GL_n have this property and that finite generation theorems are implied by it.

Necessity

Consider $k[L]$ as $k[V]$-module through restriction $\phi \mapsto \phi|L$.
Then f exists iff $k[L]^G$ is noetherian module for $k[V]^G$.

Nagata 1964 showed that if G is geometrically reductive, the finite generation theorems hold (in the general case, not just the linear one.)

Haboush 1975
'Reductive groups are geometrically reductive.'

Example $G = \mathbb{G}_a$ acting on $V = \mathbb{C}^2$ through matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Then $k[V]^G \to k[L]^G = k[L]$ hits only k, so G is not geometrically reductive and Weitzenböck does not extend from the traditional linear case to the modern general case.

Nagata uses an intermediate result, which we call
Power Reductivity
to distinguish it from other formulations of geometric reductivity. Reading $k[V]$ as $S^*(V^\vee)$ one is led to the

Property (Power Reductivity)
Let L be a cyclic k-module with trivial G-action. Let M be a rational G-module, and let φ be a G-module map from M onto L. Then there is a positive integer d such that the d-th symmetric power of φ induces a surjection:

$$(S^d M)^G \to S^d L.$$

So one passes from the geometric thinking implicit in $k[V]$ to the algebraic thinking implicit in $S^* M$. This makes all the difference when one tries to generalize from fields to arbitrary base rings.

This becomes apparent when one compares

C. S. Seshadri,
Geometric reductivity over arbitrary base,
Advances in Math. 26 (1977), no. 3, 225–274

with our preprint

Vincent Franjou, Wilberd Van Der Kallen,
Power reductivity over an arbitrary base
arXiv:0806.0787

The problem with using $k[V]$ is that it requires that V is free as a k module. Therefore Seshadri runs into the difficult issue of resolving equivariant sheaves by equivariant vector bundles. This was later treated by Bob Thomason [Adv. in Math. 65 (1987)]. But even with such help only the case of a Nagata base ring was covered.

Our point of view: Modules are not always free and this is great, but one should avoid taking their duals.

Main result

Let G be GL_n/R, the group scheme GL_n over the commutative ring R, or a Chevalley subgroup scheme of GL_n/R (based on an admissible lattice).

Theorem G is power reductive.

Idea of proof:
Go local, lift the Steinberg modules of our favorite modular proof [CPSvdK, Invent. Math. 39 (1977)], use Nakayama lemma and apply the cohomological universal coefficient theorems.

Corollary

Let G act rationally on the algebra A leaving the ideal J invariant. Every $b \in (A/J)^G$ has a power that lifts to A^G.

Corollary ['Nagata']

If R is noetherian and A is f.g, then so is A^G.

Example

Let $R = \mathbb{Z}$, $G = SL_2$ acting in its adjoint representation M with basis

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The class of H in $M/2M$ is invariant. One does not expect such a modular invariant to lift to characteristic zero, does one?

Power reductivity tells that actually some power of $H \bmod 2M \in (S^*(M/2M))^G$ must lift to $(S^*M)^G$.

Indeed $H^2 + 4XY \in (S^2M)^G$.

Note that it is essential that one allows nonfree modules when forging this link between characteristics.

Why did we care about A^G ? We hope one day to prove that not only $A^G = H^0(G, A)$, but all of $H^*(G, A)$ is finitely generated. Over fields this has now been achieved by Antoine Touzé. To get such a result over the integers say, we first need to understand the situation over $\mathbb{Z}$ a lot better. That is why.

At the moment we are at an exploratory stage, using a rather primitive kind of computer algebra to find auxiliary cocycles, in the hope of eventually lifting all cocycles employed by Touzé.

Snapshot

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(* D = X[1,3], w lifts an invariant, v lifts Dw *)
w = tensor[e[1, 1], wedge[e[1, 2], e[2, 1]]] +
 tensor[e[1, 1], wedge[e[1, 3], e[3, 1]]] -
 tensor[e[1, 2], wedge[e[2, 1], e[2, 2]]] +
 tensor[e[1, 2], wedge[e[2, 3], e[3, 1]]] -
 tensor[e[1, 3], wedge[e[2, 1], e[3, 2]]] -
 tensor[e[1, 3], wedge[e[3, 1], e[3, 3]]] +
 tensor[e[2, 2], wedge[e[2, 3], e[3, 2]]] -
 tensor[e[2, 3], wedge[e[3, 2], e[3, 3]]]
v = -tensor[e[1, 1], e[1, 1], e[1, 3]] -
 tensor[e[1, 1], e[1, 2], e[2, 3]] -
 tensor[e[1, 2], e[1, 1], e[2, 3]] -
 tensor[e[1, 1], e[1, 3], e[3, 3]] -
 tensor[e[1, 3], e[1, 1], e[3, 3]] -
 tensor[e[2, 2], e[1, 2], e[2, 3]] -
 tensor[e[1, 2], e[2, 2], e[2, 3]] -
 tensor[e[2, 3], e[1, 2], e[3, 3]] -
 tensor[e[1, 2], e[2, 3], e[3, 3]] +
 tensor[e[2, 3], e[1, 3], e[3, 2]] +
 tensor[e[1, 3], e[2, 3], e[3, 2]]
DDv = 2*tensor[e[1, 3], e[1, 3], e[1, 3]] (* survives in characteristic three *)
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Take it easy, Arjeh !