Cohomological Finite Generation for reductive groups, Luminy 2011

Wilberd van der Kallen 

 Antoine Touzé

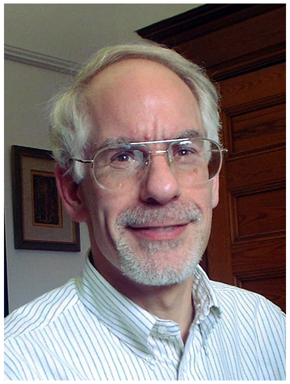
Let k be a field.
Let G be a reductive group over k
acting on a finitely generated
commutative k-algebra A.

The (CFG) Theorem (Duke Journal 2010)
$H^*(G, A)$ is a finitely generated k-algebra.
[van der Kallen conjecture $\pm$ 2003]

 Leonard Evens

Let k be a commutative noetherian ring.
Let G be a finite group
acting on a finitely generated
commutative k-algebra A.

The (CFG) Theorem of Evens (TAMS 1961)
$H^*(G, A)$ is a finitely generated k-algebra.

 Eric Friedlander and Andrei Suslin 

Let G be a finite group scheme over the field k.
Let C be a commutative noetherian k-algebra on which G acts trivially.
Let G act on a finitely generated commutative C-algebra A.

Their (CFG) Theorem (Inventiones 1997)
Say G is infinitesimal. $H^*(G, A)$ is a finitely generated C-algebra, noetherian as a module over an explicit algebra $A^G \otimes_k S(\ldots)$.

The Setting

k is the base ring. It is commutative and often noetherian.
G is *flat* affine algebraic group scheme over k.
Its coordinate ring $k[G]$ is a Hopf algebra.
A G-*module* is a comodule for the Hopf algebra $k[G]$.
It is also called a *representation*. Thus a representation is a k-module V together with a comultiplication $\Delta : V \to V \otimes_k k[G]$ which makes $G(R)$ act R-linearly on $V \otimes_k R$, functorially in R.

It is essential that G be *flat*, but V need not be flat or finitely generated. The representations form an abelian category with enough injectives.
The subspace V^G of *invariants* in V corresponds with $\mathrm{Hom}(k, V)$.
Put $H^i(G, V) = \mathrm{Ext}^i(k, V)$. Cohomology can be computed by means of the Hochschild complex $C^\bullet(V) = (V \otimes_k C^\bullet(k[G]))^G$.
There is a DGA structure on $C^\bullet(k[G]) = k[G]^{\otimes(\bullet+1)}$.

Say A is a commutative k-algebra on which G acts by algebra automorphisms. Then $A \otimes_k A \to A$ is a G-module map. Similarly, if M is an A-module on which G acts compatibly then $A \otimes_k M \to M$ is a G-module map.

Any (CFG) theorem implies one for noetherian A-modules M by considering the symmetric algebra $S_A(M)$. For instance, the (CFG) theorem of Evens also tells that $H^*(G, M)$ is a noetherian $H^*(G, A)$-module.

As $H^*(G, A)$ is graded commutative one often restricts attention to $H^{\text{even}}(G, A)$. It turns out that when $A = k$ is a field, G a finite group, the support of the module $H^*(G, M \otimes_k M^\vee)$ in (the spectrum of) $H^{\text{even}}(G, k)$ gives information on the complexity of M. The complexity measures the rate of growth of a minimal projective resolution. The theory of *support varieties* has been very successful.

Elusive

It was desirable to imitate the theory of support varieties for finite group schemes instead of finite groups. But extending the CFG theorem of Evens in that direction proved ‘surprisingly elusive’.

The way finite group schemes arise in the representation theory of reductive groups in characteristic p is as follows. Say $k = \bar{\mathbb{F}}_p$ and G is a reductive group over k. The Frobenius map raises elements of $k[G]$ to the p-th power. It defines a morphism of group schemes $F : G \to G$. The scheme theoretic kernel G_r of F^r is known as the r-th Frobenius kernel. The G_r serve as analogue of the Lie algebra of a connected Lie group. If V, W are finite dimensional G-modules, then $\mathrm{Ext}^i_G(V, W)$ is the projective limit for $r \to \infty$ of the $\mathrm{Ext}^i_{G_r}(V, W)$.

While Friedlander and Suslin were working on their (CFG) theorem, I noticed a Lemma. Something like this.

Lemma

Let G be an algebraic group (smooth) over a field and let H be a geometrically reductive subgroup scheme.
If G satisfies (CFG), so does H.

This made one wonder if (CFG) holds for GL_N over a field, in particular for large N. In that case (CFG) holds both for finite group schemes and for reductive groups (geometrically reductive by Haboush 1975). It turns out that GL_2 is much easier. In fact I can now do (CFG) for GL_2 over any noetherian base ring k.

And of course over a field of characteristic zero one has $H^{>0}(GL_N, A) = 0$, so that (CFG) is just invariant theory.

Geometric reductivity

Let us call a k-module M *geometric* if it is finitely generated projective. In that case the functor $R \mapsto M \otimes_k R$ is representable by an affine scheme, the spectrum of the coordinate ring $S_k(M^\vee)$.

For a reductive group G over a field of positive characteristic p Mumford (1965) conjectured the following:

Let $M \to k_\xi \to 0$ be exact, where M is geometric and k_ξ denotes the representation defined by the character $\xi : G \to \mathbb{G}_m = GL_1$. Then there is $n \geq 1$ so that $S^n M \to S^n k_\xi$ splits. This property is called *geometric reductivity* and was proved by Haboush (1975).

Now if k is not a field, there is a much superior notion, obtained by *not requiring* that M be geometric.

 with Vincent Franjou at Madrid, 2006, ©MFO (Author: Greuel, Gert-Martin)

We call G *power reductive* if, whenever $M \to k_\xi \to 0$ is exact, there is $n \geq 1$ so that $S^n M \to S^n k_\xi$ splits.

If k is noetherian then power reductivity implies finite generation of invariants (FG): If A is finitely generated then so is A^G. This was proved by Nagata (1964) over fields. The proof goes through here, as is clear from the exposition by Springer (1977) of Nagata's proof.

Power reductivity is preserved by change of base ring.
Descent. If $k \to R$ is faithfully flat and G_R is power reductive, then so is G_k. A similar statement holds for (CFG).
If N is normal subgroup so that N and G/N are power reductive, then so is G.

If G is power reductive and I is an invariant ideal in A, then for every $b \in (A/I)^G$ there is an $n \geq 1$ so that b^n lifts to A^G.

Chevalley groups

Chevalley groups are power reductive: Say $G_{\mathbb{Z}}$ is a connected split reductive algebraic $\mathbb{Z}$-group . Then it is power reductive. [Easy! Go local on Spec($\mathbb{Z}$), then use Nakayama Lemma and Universal Coefficient Theorem to lift the Cline-Parshall-Scott proof of Haboush's Theorem, based on Kempf vanishing and Steinberg modules.]

Example. If $G_{\mathbb{Z}} = SL_N$ acts on a flat $\mathbb{Z}$-algebra A, then one may take a prime number p and put $I = pA$. Thus 'modular invariants lift up to a power'.

Also note that $(A/pA)^{G_{\mathbb{Z}}} = (A/pA)^{G_{\mathbb{Z}/p\mathbb{Z}}}$. More generally, if $k \to R$ is a map of commutative rings, and M is a G_R-module, then it may also be viewed as G-module via $M \otimes_R R[G] = M \otimes_k k[G]$ and $H^i(G_R, M) = H^i(G, M)$.

If G is *not* power reductive, for instance $G = \mathbb{G}_a$ over a field, take a counter example $M \to k \to 0$ with M finitely generated. Then $S_{S(M)}(S(k))$ is finitely generated, but $(S_{S(M)}(S(k)))^G$ is not (Exercise). So power reductivity is *equivalent* to (FG) over a noetherian base ring k.

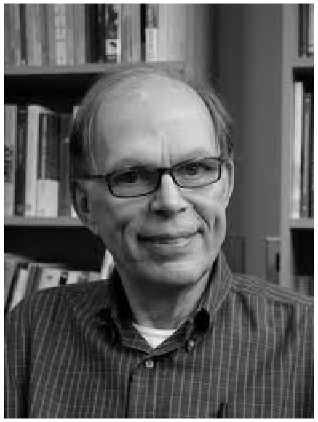
Problem 14 of Hilbert is about domains. That is much harder. The example $S_{S(M)}(S(k))$ is no domain.

The upshot is that over a field (FG) is equivalent to (CFG).

Over other noetherian base ring we know much less about (CFG). Say k is noetherian and $G = G_k$ with $G_{\mathbb{Z}}$ a Chevalley group scheme as above.

Assume as always that the commutative algebra A is finitely generated over k, with rational action on A of G. Further, let M be a noetherian A-module again with compatible G-action.
An abelian group L has *bounded torsion* if there is an $n \geq 1$ with $nL_{\text{tors}} = 0$.

- Every $H^m(G, M)$ is a noetherian A^G-module.
- If $H^*(G, A)$ is a finitely generated k-algebra, then $H^*(G, M)$ is a noetherian $H^*(G, A)$-module.
- $H^*(G, A)$ is a finitely generated k-algebra iff it has bounded torsion.

And if you know about the Grosshans 
graded deformation gr A of A, you may add

- $H^*(G, \text{gr}\, A)$ is a finitely generated k-algebra.

Notice that $H^1(\mathbb{G}_a, \mathbb{F}_p)$ is not finite dimensional.
That is why we restricted to Chevalley groups.
Maybe power reductivity would do.

Why conjecture (CFG)?

Now why did (CFG) become a conjecture? Not just because it holds for GL_2 and would be nice in general. It took me several years.

The work of Friedlander and Suslin seems to prepare for more than what they use it for. They construct cohomology classes for GL_N and then only use them infinitesimally.

Another reason to start believing (CFG) is that (CFG) has implications that happen to pan out, despite them looking unreasonable to me (initially).

Here is a recent one, based on joint work with Srinivas

Let k be noetherian, $G = SL_N$. Recall that one has Schur modules or costandard modules $\nabla(\lambda) = \Gamma(G/B, \mathcal{L}_\lambda)$ with highest weight λ. Say a representation V has *good Grosshans filtration* if $H^i(G, V \otimes_k \nabla(\lambda))$ vanishes for $i > 0$ and λ dominant.

Assume again that the commutative algebra A is finitely generated over k, with rational action on A of G. Further, let M be a noetherian A-module with compatible G-action.

Theorem

Recall $G = SL_N$. If A has a good Grosshans filtration, then M has a finite resolution $0 \to M \to N_0 \to \cdots \to N_d \to 0$ by noetherian A-modules with good Grosshans filtration.

If one drops 'noetherian' then this is a corollary of the fact that (CFG) holds in the bounded torsion case.
So without 'noetherian' it also holds for other Dynkin types.

The proof of the Theorem uses 'Characteristic free resolution of the ideal of the diagonal', which is a method borrowed from a computation of equivariant K-groups of Grassmannians.

Strands

The proof of (CFG) for GL_N over a field depends on many ingredients. One needs the results of Friedlander and Suslin in explicit form, Grosshans graded deformation, an inseparabilty lemma of Mathieu (which is where it becomes very nonconstructive), the characteristic free ‘resolutions of the diagonal’ for products of Grassmannians, and the all important new ingredient: the universal cohomology classes of Touzé. See the paper in the Duke Journal, which has expository parts.

The universal cohomology classes are constructed in the setting of strict polynomial bifunctors, invented by Franjou and Friedlander. This setting models a stable (*i.e.* $N \to \infty$) version of GL_N-cohomology.

One studies Ext groups in the category of strict polynomial bifunctors. The main problem ($\pm$ 2001) is to produce a family of cohomology classes that sufficiently enriches the family constructed by Friedlander and Suslin, who used functors of one variable only.

Back in the GL_N-cohomology setting the *lifted classes* $c[m]$ of Antoine Touzé are characterized by

- $c[1] \in H^2(GL_N, \mathfrak{gl}_N^{(1)})$ is nonzero,

- For $m \geq 1$ the class $c[m] \in H^{2m}(GL_N, \Gamma^m(\mathfrak{gl}_N^{(1)}))$ lifts $c[1] \cup \cdots \cup c[1] \in H^{2m}(GL_N, \bigotimes^m(\mathfrak{gl}_N^{(1)}))$.

Taking his cue from the Cartan Seminar of 1954/55, Antoine Touzé starts with the Frobenius twist of a bar resolution of a bar resolution of a symmetric algebra functor. Troesch has invented a construction of an injective resolution of a Frobenius twist of a tensor product of symmetric powers. Antoine Touzé applies the Troesch construction componentwise to the iterated bar resolution, in the hope of getting a double complex in which appropriate cochains can be located.

A miracle is needed because the Troesch construction is not functorial, so that it seems a bit optimistic to expect a double complex.

To perform the miracle Antoine Touzé changes the rules by inventing a new category that is just rich enough to contain the iterated bar resolution, but so special that the Troesch construction is functorial on it.

Nowadays he has a different proof based on a general formality theorem for Frobenius twists.

Bibliography

- K. Akin, D. Buchsbaum, J. Weyman, Schur functors and Schur complexes, Adv. in Math. 44 (1982), 207–278.

- Nicolas Bourbaki, Éléments de mathématique. (French) Algèbre. Chapitre 10. Algèbre homologique. Masson, Paris, 1980. vii+216 pp.

- L. Evens, The cohomology ring of a finite group, Trans. Amer. Math. Soc. 101 (1961), 224–23.

- E. M. Friedlander, A. A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997), 209–270.

- Vincent Franjou and Wilberd van der Kallen, Power reductivity over an arbitrary base, Documenta Mathematica, Extra Volume Suslin (2010), pp. 171-195.

- F. D. Grosshans, Contractions of the actions of reductive algebraic groups in arbitrary characteristic, Invent. Math. 107 (1992), 127–133.

- J.-C. Jantzen, Representations of Algebraic Groups, Mathematical Surveys and Monographs vol. 107, Amer. Math. Soc., Providence, 2003.

- M. Levine, V. Srinivas, J. Weyman, K-Theory of twisted Grassmannians, K-Theory 3 (1989), 99–121.

- M. Nagata, Invariants of a group in an affine ring, J. Math. Kyoto Univ. 3 (1963/1964), 369–377.

- T. A. Springer, Invariant theory. Lecture Notes in Mathematics, 585. Springer-Verlag, Berlin-New York, 1977.

- V. Srinivas, W. van der Kallen, Finite Schur filtration dimension for modules over an algebra with Schur filtration, Transformation Groups, Vol. 14, No. 3, 2009, pp. 695-711. DOI: 10.1007/S00031-009-9054-0

- A. Touzé, Universal classes for algebraic groups, Duke Mathematical Journal, Vol. 151, No. 2, 2010, pp. 219-249. DOI: 10.1215/00127094-2009-064

- Antoine Touzé and Wilberd van der Kallen, *Bifunctor cohomology and Cohomological finite generation for reductive groups*, Duke Mathematical Journal, Vol. 151, No. 2, 2010, pp. 251-278. DOI: 10.1215/00127094-2009-065

- W. van der Kallen, Cohomology with Grosshans graded coefficients, In: Invariant Theory in All Characteristics, Edited by: H. E. A. Eddy Campbell and David L. Wehlau, CRM Proceedings and Lecture Notes, Volume 35 (2004), 127-138.