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Cohomological Finite Generation and the Identity Correspondence

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CFG Conjecture (2004)

Theorem (Touzé 2010). Let G be a reductive linear algebraic group defined over a field k. Let G act on a finitely generated commutative k-algebra A.
Then $H^*(G, A)$ is a finitely generated k-algebra.

Corollaries/precursors of CFG

If k has characteristic zero: Representations of reductive G are completely reducible.

Hilbert problem 14. $H^*(G, A) = A^G$ is finitely generated.

If $R \to A$ is an equivariant surjection, then $R^G \to A^G$ is surjective. So it suffices to look at action of G on a polynomial ring R, respecting its grading. Classical invariant theory does this.

Theorem (Evens 1961) CFG holds if G is a finite group.

Principle In a spectral sequence of modules, if the starting page is noetherian, then all pages are noetherian and the spectral sequence stops (degenerates).

Reductivity in positive characteristic

Let k have characteristic $p > 0$.

FvdK: Call $f : A \to R$ *power surjective* if for every $r \in R$ there is $n \geq 1$ with $r^n \in f(A)$.

Haboush (1975) has shown that if the equivariant $A \to R$ is power surjective, then so is $A^G \to R^G$.

We call this *power reductivity* of G. Over our field it is equivalent to *geometric reductivity*, which only considers certain equivariant maps between polynomial rings.

Nagata (1964) had shown that power reductivity implies that $H^0(G, A) = A^G$ is a finitely generated algebra. These results remain valid for reductive group schemes over a noetherian base ring.

Setting: affine group schemes flat over affine base.

Power reductivity is preserved by base change. It satisfies faithfully flat descent and a local global principle.

Infinitesimal group schemes

Let G be an infinitesimal group scheme. Its coordinate ring $k[G]$ is a local artinian algebra. Note that although G is no reductive linear algebraic group, it is still power reductive.

Theorem (Friedlander–Suslin 1997) Let G be an infinitesimal group scheme over k acting on a finitely generated commutative k-algebra A. Then $H^*(G, A)$ is a finitely generated k-algebra.

Strict polynomial functors

If F is a functor and a group G acts on X, then it also acts on FX. If F is a strict polynomial functor and X is a representation of an algebraic group G, then so is FX.
Example: $FX = S^4(X \otimes k^5)$

Frobenius kernel

Example. Let G be a reductive linear algebraic group defined over $\mathbb{F}_p$. Let $F : G \to G$ be the Frobenius homomorphism. The r-th Frobenius kernel G_r is the scheme theoretic kernel of F^r. One has an exact sequence in the fppf topology

$$1 \to G_r \longrightarrow G \xrightarrow{F^r} G \to 1$$

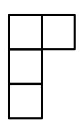
It turns out that the cohomology of G ('global'), is closely related to that of the G_r ('local').

For finite dimensional G modules M, N one may compute $\mathrm{Ext}^i_G(M, N) = H^i(G, \mathrm{Hom}_k(M, N))$ as the inverse limit of the $\mathrm{Ext}^i_{G_r}(M, N)$.

Costandard modules

Let us fix G, B, T, U, defined over $\mathbb{F}_p$. Think $G = \mathsf{GL}_n$.

Let $\mathcal{L}$ be an equivariant line bundle on the flag variety G/B. If $\Gamma(G/B, \mathcal{L}) \neq 0$, then we write this indecomposable G-module as $\nabla(\lambda)$, where λ is its highest weight. We call it the costandard module of highest weight λ.
One then has Kempf vanishing: $H^i(G/B, \mathcal{L})$ vanishes for $i > 0$ and 'therefore' $H^i(G, \nabla(\lambda))$ vanishes for $i > 0$.

If $\nabla(\lambda)$ is a polynomial representation of GL_n then one also calls it a Schur module. Then λ corresponds with a Young diagram like .

ht
Fix an additive height function ht on the weight lattice that takes value two on simple roots and value zero on weights of one dimensional representations of G, like $\det : \mathrm{GL}_n \to \mathbb{G}_\mathrm{m}$.

Grosshans filtration
If M is a G-module then $M_{\leq i}$ is the largest G-submodule all whose weights λ have height $\mathrm{ht}(\lambda)$ at most i.

We say that M has good filtration if the associated graded $\mathrm{gr}(M)$ of its Grosshans filtration is a direct sum of costandard modules.
Then $H^i(G, M) = 0$ for $i > 0$.

Grosshans graded deformation $\mathcal{A}$
Grosshans (1992): If A is a finitely generated commutative k-algebra then so is $\mathrm{gr}\, A$ and $\mathrm{gr}\, A$ is a flat deformation of A.

Suppose $G = \mathsf{GL_n}$. Let A be a finitely generated k algebra with good filtration. Let M be a finitely generated A module with $A \otimes M \to M$ equivariant.
Then we have the following corollary/precursor of CFG

Theorem (Srinivas-vdK 2009) There is a finite resolution

$$0 \to M \to N_0 \to \cdots \to N_s \to 0$$

by modules N_j with good filtration and the $H^i(G, M)$ are finitely generated A^G modules.

Example: product of Grassmannians.

Our $G = \mathsf{GL_n}$ acts on the Grassmannian $\mathsf{Gr}(s)$ of s dimensional subspaces of k^n. Let

$$X = \mathsf{Gr}(s_1) \times \cdots \times \mathsf{Gr}(s_r)$$

with Cox ring

$$A = \bigoplus_{\mathcal{L} \in \mathsf{Pic}(X)} \Gamma(X, \mathcal{L}).$$

It has a good filtration [Wang Jian-Pan 1981].
If $\mathcal{M}$ is an equivariant coherent sheaf on X, consider

$$M = \bigoplus_{\mathcal{L} \in \mathsf{Pic}(X)_{\geq 0}} \Gamma(X, \mathcal{M} \otimes \mathcal{L}).$$

Our theorem should apply.

Resolution of the diagonal

Srinivas: Apply the identity correspondence to $\mathcal{M}$

$$\mathcal{M} = I\mathcal{M}$$

and study $\mathcal{M}$ by resolving the sheaf $\mathcal{O}_{\Delta}$ of the graph Δ of I.

Compare M. Levine, V. Srinivas, J. Weyman,
K-Theory of twisted Grassmannians (1989)

Say $X = \mathrm{Gr}(s)$. We have the tautological vector bundle $\mathcal{S}$ on X with fiber V at V. We have the quotient bundle $\mathcal{Q}$ with fiber k^n/W at W. Given s dimensional subspaces V, W we have a natural map $V \to k^n/W$. It vanishes if and only if $V = W$. So we get a description of the diagonal Δ of $X \times X$ as the locus where a certain morphism of vector bundles $\mathrm{pr}_1^* \mathcal{S} \to \mathrm{pr}_2^* \mathcal{Q}$ vanishes.

Dualizing one gets an exact sequence

$$\mathcal{E} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{\Delta} \to 0$$

where $\mathcal{E} = \mathcal{S} \boxtimes \mathcal{Q}^\vee := \mathrm{pr}_1^* \mathcal{S} \otimes \mathrm{pr}_2^* \mathcal{Q}^\vee$. The Koszul complex

$$0 \to \bigwedge^d \mathcal{E} \to \cdots \to \mathcal{E} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{\Delta} \to 0$$

of $\mathcal{E} \to \mathcal{O}_{X\times X}$ provides a resolution by vector bundles of the sheaf $\mathcal{O}_{\Delta}$. The identity correspondence

$$I : \mathcal{M} \mapsto \mathrm{pr}_{1*}(\mathcal{O}_{\Delta} \otimes \mathrm{pr}_2^* \mathcal{M})$$

can therefore also be described in terms of the $\wedge^t\mathcal{E}$.
But Akin–Buchsbaum–Weyman (1982) tell us that the $\wedge^t\mathcal{E}$ can be filtered with quotients that are exterior tensor products of Schur functors applied to $\mathcal{S}$ and coSchur functors applied to $\mathcal{Q}^\vee$.

Schur functors are associated with Young diagrams.
In our convention the Schur functor associated with $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ sends V to $V \wedge V$ and the Schur functor associated with $\square\square\square$ sends V to S^3V. (Some papers use the conjugate convention.)

We are happy about the Schur functors because they produce costandard modules. We are less happy about the coSchur functors, so we place them at the second factor X where they get tensored with $\mathcal{M}$. Then pr_{1*} integrates this junk out and leaves a finite dimensional representation. Those we can handle [Friedlander–Parshall 1986].

$$\mathrm{pr}_{1*}((nice \boxtimes other) \otimes \mathrm{pr}_2^* \mathcal{M}) = nice \otimes \mathrm{pr}_*(other \otimes \mathcal{M})$$

$\square$

Hull

Let A be a finitely generated k-algebra on which $G = \mathsf{GL_n}$ acts. Grosshans (1992) studied an embedding of finitely generated algebras $\mathsf{gr}\, A \hookrightarrow \mathsf{hull}(\mathsf{gr}\, A)$, where $\mathsf{hull}(\mathsf{gr}\, A)$ is a direct sum of costandard modules and $(\mathsf{gr}\, A)^U = (\mathsf{hull}(\mathsf{gr}\, A))^U$.

Mathieu (1990) observed that there is an $r \geq 1$ so that $\mathsf{gr}\, A$ contains all p^r-th powers of elements of $\mathsf{hull}(\mathsf{gr}\, A)$. Take such r.

Combining the computations of [Friedlander–Suslin 1997] with [Srinivas–vdK 2009] one concludes that the Hochschild–Serre spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \mathsf{gr}\, A)) \Rightarrow H^{i+j}(G, \mathsf{gr}\, A)$$

has a finitely generated abutment $H^*(G, \mathsf{gr}\, A)$.

Touzé classes

To get to CFG for A rather than for $\mathrm{gr}\,A$, Touzé constructs cohomology classes that allow to conclude that

$$\mathrm{res} : H^*(G,R) \to H^*(G_r,R)$$

makes $H^0(G/G_r, H^*(G_r,R))$ into a noetherian module over $H^*(G,R)$ whenever R is a finitely generated k algebra.

The Touzé classes live in Ext^i groups of strict polynomial bifunctors and have since been instrumental in getting a better understanding of the effect of Frobenius twist on Ext groups of strict polynomial functors. Schur functors and coSchur functors and Frobenius twist are such functors.

We want to understand the difference between $\mathrm{Ext}^*_{\mathcal{P}(d)}(P,Q)$ and $\mathrm{Ext}^*_{\mathcal{P}(dp)}(P^{(1)},Q^{(1)})$ for strict polynomial functors P, Q of degree d.
Put

$$E_1 := \bigoplus_{i=0}^{p-1} k_{2i},$$

where k_j denotes the one dimensional $\mathbb{G}_\mathrm{m}$ module of weight j.
Theorem (Touzé, Chałupnik, 2013, 2014)

$$\mathrm{Ext}^n(P^{(1)},Q^{(1)}) \cong \bigoplus_{i+j=n} \mathrm{Ext}^i\left(P(?),Q(?\otimes E_1)^j\right),$$

where $Q(V\otimes E_1)^j$ is the weight j component for the $\mathbb{G}_\mathrm{m}$ action.
Example:

$$S^2(V\otimes E_1)^4 = \Big((V\otimes k_0)\otimes(V\otimes k_4)\Big)\oplus S^2(V\otimes k_2)$$

Thank You !

Crash course on Strict polynomial (bi)functors.

Divided power

Let $\Gamma^d M := (M^{\otimes d})^{\mathfrak{S}_d} = (S^d(M^\vee))^\vee$ denote the d-th divided power of a finite dimensional vector space M.
Γ^d is a coSchur functor.

Schur algebra

By Schur (1901) a module for the Schur algebra $S(n,d) := \Gamma^d \mathrm{Hom}(k^n, k^n)$ gives rise to a polynomial representation of GL_n.
Schur functors and coSchur for Young diagrams with d boxes provide examples of polynomial representations. (Apply the functor to the defining representation k^n of GL_n.)

Representations of a k-linear category

One thinks of a k-linear category as a multi-object algebra. Indeed an algebra A is a k-linear category with one object. An A-module M is a k-linear functor from A to k-Mod. $\mathsf{Rep}\,A$ is the category of A-modules.

Schur category

The *Schur category* $S(d)$ of degree d is the k-linear category whose objects are finite dimensional vector spaces and whose morphisms are given by

$$\mathsf{Hom}_{S(d)}(V,W) := \Gamma^d\,\mathsf{Hom}_k(V,W).$$

Strict polynomial (bi)functors

Strict polynomial functors of degree d are objects of $\mathcal{P}(d) := \operatorname{Rep} S(d)$. They restrict to representations of Schur algebras $S(n,d) = \operatorname{Hom}_{S(d)}(k^n, k^n)$.
One often confuses an element P of $\operatorname{Rep} S(d)$ (Pirashvili model of strict polynomial functor) with the functor that sends a linear map f to $P(f^{\otimes d})$ (Friedlander–Suslin model).

Schur functors and coSchur for Young diagrams with d boxes provide examples of strict polynomial functors of degree d.

Strict polynomial bifunctors are bimodules for $S(d)$.

If $n \geq d$, then restriction is an equivalence of categories $\mathcal{P}(d) = \operatorname{Rep} S(d) \to \operatorname{Rep} S(n,d)$ and one has $\operatorname{Ext}^*_{\mathcal{P}(d)}(P,Q) \cong \operatorname{Ext}^*_{\mathsf{GL}_n}(P(k^n), Q(k^n))$.

Frobenius twist

The Frobenius twist $V^{(1)}$ of a representation V of GL_n is obtained by precomposing with $F : \mathsf{GL}_n \to \mathsf{GL}_n$.

Its strict polynomial functor analogue is precomposition by the *Frobenius twist functor* $I^{(1)} := \ker(S^p \to \Gamma^p)$.

$I^{(1)}$ is another important example of a strict polynomial functor. It has degree p.

Put $P^{(1)} := P \circ I^{(1)}$.

Example

Let $G = \mathbb{G}_{\mathsf{a}}$ be the Lie group $\mathbb{C}$ with addition as operation.

Let $t \in G$ act on $A = \mathbb{C}[X, Y, Z]/(XZ)$ by
$X \mapsto X, \quad Y \mapsto Y + tX, \quad Z \mapsto Z$.
Then A^G contains X, Z and $Y^i Z$ for $i \geq 1$, and A^G is not finitely generated. This is an awful lot simpler than the famous Nagata counterexample from 1959. But we have changed the rules and A is no polynomial ring. Not even a domain.

This also gives the standard example showing that $G = \mathbb{G}_{\mathsf{a}}$ fails power reductivity: If I is the ideal generated by X in A, then no power of $Y \in (A/I)^G$ lifts to A^G.