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Cohomological Finite Generation and Bifunctors

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Invariant theory

P. Gordan 1868, J.f.d. reine u. angew. Math., 69, 323–354.

Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.

In modern language: $G = SL_2(\mathbb{C})$ as algebraic group.

$G \curvearrowright V := \mathbb{C}^2, \quad \mathbb{C}[V] = \mathbb{C}[X,Y], \quad W_d := \mathbb{C}[V]_d,$

$W_2 = \{aX^2 + bXY + cY^2\}, \quad \mathbb{C}[W_2] = \mathbb{C}[a,b,c],$

$b^2 - 4ac \in \mathbb{C}[W_2]^G$ an *invariant* (= fixed point).

Gordan: $\mathbb{C}[W_d]^G$ is finitely generated (f.g.) as a $\mathbb{C}$-algebra.

Hilbert 1890

$G = SL_n(\mathbb{C})$ acting algebraically on some finite dimensional complex vector space V. Here 'algebraically' means the action is given by polynomials: For each $v \in V$ there is a polynomial f_v in the matrix entries of $g \in G$ with coefficients in V so that $g \cdot v = f_v(g)$.

Example: The above action of $G = SL_2(\mathbb{C})$ on W_2.

Then Hilbert shows nonconstructively that $\mathbb{C}[V]^G$ is finitely generated as a $\mathbb{C}$-algebra.

Examples Consider the action of $G = GL_n(\mathbb{C})$ by conjugation on the vector space $V = M_n(\mathbb{C})$ of $n \times n$ matrices. So $g \in G$ sends $m \in V$ to gmg^{-1}. Then $\mathbb{C}[V]^G$ is generated by the coefficients c_i of the characteristic polynomial
$\det(m - \lambda I) = c_0 + c_1\lambda + \cdots + c_n\lambda^n$.

Next let G be the group of permutations of the n variables in the polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$. Then $\mathbb{C}[X_1, \ldots, X_n]^G = \mathbb{C}[p_1, \ldots, p_n]$, where
$p_i = X_1^i + \cdots + X_n^i$.

Encouraged by an incorrect claim of Maurer, Hilbert asked in his fourteenth problem if this finite generation of invariants is a general fact about actions of algebraic Lie groups on domains of finite type over $\mathbb{C}$.

A counterexample of Nagata (1959) showed this was too optimistic.

By then it was understood that finite generation of invariants does hold for compact connected real Lie groups like orthogonal groups (cf. Hurwitz 1897). Hurwitz considers a compact group K with Haar measure dk and introduces the method of averaging. $K \curvearrowright V$ linear. Get linear equivariant retract $V \to V^K$ from

$$v \mapsto \frac{\int_K kv\, dk}{\int_K dk}.$$

Finite generation also holds for the complexifications of compact Lie groups, also known as the connected reductive complex algebraic Lie groups (Weyl 1926).

Finite groups have been treated by Emmy Noether (1926), so connectedness may be dropped. (Algebraic Lie groups have finitely many connected components.) She worked over an arbitrary ground field.

Mumford (GIT, 1965) needed finite generation of invariants for reductive algebraic groups over fields of arbitrary characteristic in order to construct moduli spaces.

Say k is an infinite field and $G = SL_n(k)$ is acting algebraically on some finite dimensional k-vector space V.
Then Mumford needs in particular that $k[V]^G$ is finitely generated as a k-algebra.

In his book Geometric Invariant Theory (1965) Mumford introduced a condition, often referred to as *geometric reductivity*. He conjectured it to be true for reductive algebraic groups and he conjectured it implies finite generation of invariants. These conjectures were confirmed by Haboush (1975) and Nagata (1964) respectively. Nagata treated any commutative algebra of finite type over the base field, not just domains. We adopt this generality. It rather changes the problem of finite generation of invariants.

The proof of Nagata was actually based on a property that Franjou and the speaker call ‘power reductivity’ (2010).

We call a map of commutative k-algebras $\phi : A \to B$ *power surjective* if for every $b \in B$ there is $n \geq 1$ so that $b^n \in \phi(A)$.

We call G *power reductive* if taking invariants preserves power surjectivity.

Thus, if a power reductive G acts algebraically on commutative k-algebras A, B, and $\phi : A \to B$ is a power surjective equivariant map, then $A^G \to B^G$ is also power surjective.

Power reductivity is the superior notion when the base ring is no longer assumed to be a field. It has better base change properties than geometric reductivity. If the base ring is noetherian the argument of Nagata goes through.

Example

Let $G = \mathbb{G}_a$ be the Lie group $\mathbb{C}$ with addition as operation.

Let $t \in G$ act on $A = \mathbb{C}[X, Y, Z]/(XZ)$ by
$X \mapsto X, \quad Y \mapsto Y + tX, \quad Z \mapsto Z$.
Then A^G contains X, Z and Y^iZ for $i \geq 1$, and A^G is not finitely generated. This is an awful lot simpler than the famous Nagata counterexample from 1959. But we have changed the rules and A is no polynomial ring. Not even a domain.

This also gives the standard example showing that $G = \mathbb{G}_a$ fails power reductivity: If I is the ideal generated by X in A, then no power of $Y \in (A/I)^G$ lifts to A^G.

Such failure of lifting is what the cohomology group $H^1(G, I)$ is about. So we naturally end up studying cohomology when looking at invariant theory. One has $A^G = H^0(G, A)$ and the $H^i(G, -)$ are the derived functors of the fixed point functor $(-)^G$.

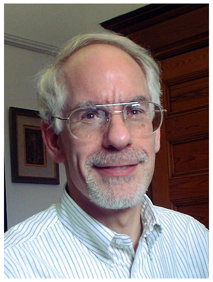
Let us say that G satisfies the cohomological finite generation property (CFG) if, whenever G acts on a commutative algebra A of finite type over k, the cohomology algebra $H^*(G, A)$ is also finitely generated over k.

Evens (1961) proved that finite groups have (CFG) and this has been the starting point of the theory of *support varieties*.

In this theory one exploits a connection between the rate of growth of a minimal projective resolution and the dimension of a ‘support variety’, which is a subvariety of the spectrum of $H^{\text{even}}(G,k)$.

People working in representation theory of algebraic groups were eager to join this activity. Thus they had to show that the result of Evens extends from finite groups to finite group schemes. (An algebraic group scheme G is called finite if its coordinate ring $k[G]$ is a finite dimensional vector space.)

This turned out to be *surprisingly elusive* (Friedlander Suslin 1997). Eric Friedlander and A. Suslin had to invent a new representation theory, the *strict polynomial functors*, in order to construct universal cohomology classes that enabled them to bring some Hochschild–Serre spectral sequences under control. Their representation theory uses the algebras $S(n,d)$ introduced by I. Schur in his 1901 thesis and named *Schur algebras* by J. A. Green in 1980.



Eric

Andrei

The $S(n,d)$-modules correspond with polynomial representations, homogeneous of degree d, of GL_n. The setting of Friedlander and Suslin captures $S(n,d)$ for all n simultaneously. Intuitively one thus finds the behavior as $n \to \infty$.

Now the speaker had noticed that if one could show that GL_n has (CFG) for large n, then it would follow that finite group schemes have (CFG). I could soon prove (2004) that GL_2 has (CFG), but 2 is not large.

Then I started to find corollaries to (CFG) that seemed wrong. So the game became to disprove the corollaries. This was a big failure. Instead of disproving them, I started to prove more and more cases. Thus it became my *conjecture* that GL_n has (CFG) (when the base ring is a field).

To follow the strategy of Friedlander and Suslin and prove my conjecture, more universal cohomology classes were needed.

These universal cohomology classes were constructed by Antoine Touzé (2010) in the setting of strict polynomial bifunctors, invented by Franjou and Friedlander (2008). This setting models a stable (*i.e.* $N \to \infty$) version of GL_N-cohomology, with coefficients like $\mathrm{Hom}(\wedge^3(k^N), S^3(k^N))$ or $\Gamma^m(\mathfrak{gl}_N^{(1)})$.

Touzé

van der Kallen/Franjou

©MFO 2006 (Author: Greuel, Gert-Martin)

One studies Ext groups in the category of strict polynomial bifunctors. The main problem (± 2001) is to produce a family of cohomology classes that sufficiently enriches the family constructed by Friedlander and Suslin, who used functors of one variable only.

Back in the GL_N-cohomology setting the *lifted classes* $c[m]$ of Antoine Touzé are characterized by

- $c[1] \in H^2(GL_N, \mathfrak{gl}_N^{(1)})$ is nonzero,

- For $m \geq 1$ the class $c[m] \in H^{2m}(GL_N, \Gamma^m(\mathfrak{gl}_N^{(1)}))$ lifts
  $c[1] \cup \cdots \cup c[1] \in H^{2m}(GL_N, \otimes^m(\mathfrak{gl}_N^{(1)}))$.

Taking his cue from the Cartan Seminar of 1954/55, Antoine Touzé starts with the Frobenius twist of a bar resolution of a bar resolution of a symmetric algebra functor. Troesch (2005) has invented a construction of an injective resolution of a Frobenius twist of a tensor product of symmetric powers. Antoine Touzé applies the Troesch construction componentwise to the iterated bar resolution, in the hope of getting a double complex in which appropriate cochains can be located.

A miracle is needed because the Troesch construction is not functorial, so that it seems a bit optimistic to expect a double complex.

To perform the miracle Antoine Touzé changes the rules by inventing a new category that is just rich enough to contain the iterated bar resolution, but so special that the Troesch construction is functorial on it.

Nowadays he has a different proof based on a general ‘formality’ theorem for Frobenius twists (Touzé, Chałupnik), inspired by a paper of Franjou–Friedlander–Scorichenko–Suslin. According to Chałupnik one has the marvelous formula

$$H^n_{\mathcal{P}}(B(-_1^{(1)}, -_2^{(1)})) \cong \bigoplus_{i+j=n} H^i_{\mathcal{P}}(B(-_1, -_2 \otimes E_1)^j).$$

Here B is a bifunctor, E_1 is the p-dimensional $\mathbb{G}_m$-module $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, I^{(1)})$ computed by Friedlander and Suslin, $B(\cdots)^j$ is the component of weight j, . . .

Chałupnik shows that the decomposition actually occurs in a derived category of multifunctors and that the projectors in the decomposition can be expressed in terms of the classes of Touzé.

Chałupnik considers the total derived functor K_1^r of the right Kan extension of precomposition by Frobenius twist. Precomposing this K_1^r with Frobenius twist [sic!] yields a representable functor, represented by an object that breaks up ("formality").