Addendum to: A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices, by V. B. Mehta and Wilberd van der Kallen

Wilberd van der Kallen

March 21, 2008

1 Introduction

Theorem 3.8 of [2] implies that a certain Frobenius splitting of $G \times^B \mathfrak{b}$ is compatible with $G \times^B \mathfrak{n}_P$, where $\mathfrak{n}_P$ denotes the Lie algebra of the unipotent radical of a certain kind of standard parabolic subgroup P of $G = Gl_n$. In Exercise 5.1.E.6 of their book [1] Brion and Kumar ask to prove this same fact for any standard parabolic subgroup. Their comments 5.C suggest that they thought this has already been done in [2]. However, in [2] we did not need all standard parabolic subgroups and we only treated a class that is slightly easier. Let us now do the exercise, by discussing the necessary modifications.

1.1 A partial order.

We simplify the partial order of [2, 3.1]. We put a partial order on the set $\mathcal{I} = [1, n] \times [1, n]$ which indexes the coordinates on $\mathfrak{g}$. We declare that

$$(i, j) \leq (r, s) \iff (i \geq r \text{ and } j \leq s)$$

If S is an ideal for this partial order, i.e. if $(i, j) \leq (r, s)$ and $(r, s) \in S$ imply $(i, j) \in S$, then we define $\mathfrak{b}[S]$ to be the subspace of $\mathfrak{b}$ consisting of the matrices X with $X_{ij} = 0$ for $(i, j) \in S$. One easily sees that such a subspace is an ideal, and all ideals of $\mathfrak{b}$ arise this way. Note that ideals in

$\mathfrak{b}$ are B invariant. Let us agree to use the notation $\mathfrak{b}[S]$ only when S is an ideal for the partial order. We will find a Frobenius splitting for all $G \times^B \mathfrak{b}[S]$ simultaneously.

We must argue a little differently than in [2, 3.7]. In particular, we can not use [2, Lemma 3.3] now.

1.2 Start of proof.

We argue by induction on the size of S to show that specialization leads to the formulas indicated in [2, 3.7], but we will go in the other direction to prove that one has Frobenius splittings. The formula for $\sigma[S]$ is by definition correct when $\mathfrak{b}[S]$ equals $\mathfrak{b}$. (Note that in this case $i > j$ for $(i, j) \in S$ so that $\delta_r[S]$ vanishes for $r \leq n$.) Therefore let us now assume S contains a maximal element (s, t) with $s \leq t$. We assume the formulas true for $S' = S - \{(s, t)\}$. Put $r = s + n - t$. For $(g, X) \in U^- \times \mathfrak{b}[S']$ we first claim that with $M = X + \delta_r[S']$, the determinant $\det((gMg^{-1})_{\leq r, \leq r})$, which is of degree one in X_{st}, is divisible by X_{st}. Now this can be checked by putting X_{st} equal to zero and showing that the rank of $M_{\leq r, \leq n}$ becomes strictly less than r. Indeed $M_{\leq r, \leq n}$ is a block matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & X_{st} & \delta \\ 0 & 0 & \epsilon \end{pmatrix}$$

which becomes

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \delta \\ 0 & 0 & \epsilon \end{pmatrix}$$

when you put X_{st} equal to zero. The submatrix

$$\begin{pmatrix} \alpha & \beta \end{pmatrix}$$

has rank at most $s - 1$ and the submatrix

$$\begin{pmatrix} \gamma \\ \delta \\ \epsilon \end{pmatrix}$$

has rank at most $n - t$, so together the rank is at most $r - 1$ indeed. We may use X_{st} as the f of [2, 3.5], at least over the open subset U^- of

G/B. As $U^- \times \mathfrak{b}[S]$ is dense in $G \times^B \mathfrak{b}[S]$, the hypotheses for the residue construction are satisfied and we only need to check that it replaces the factor $\det((g(X+\delta_r[S'])g^{-1})_{\leq r,\leq r})$ in the product for $\sigma[S']$ by the factor $\det((g(X+\delta_r[S])g^{-1})_{\leq r,\leq r})$. Indeed one must put X_{st} equal to zero in the regular function

$$\det((g(X+\delta_r[S'])g^{-1})_{\leq r,\leq r})/X_{st}$$

And this gives the same as putting X_{st} equal to zero in $\det((g(X+\delta_r[S])g^{-1})_{\leq r,\leq r}$. The rest of the proof proceeds as before.

Note that we are dealing here with a residually normal crossing situation in the sense of [3].

References

[1] M. Brion and S. Kumar, Frobenius Splitting Methods in Geometry and Representation Theory, Birkhäuser Boston 2005.

[2] V.B. Mehta and W. van der Kallen, A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices, Compositio Math. 84 (1992), 211–221.

[3] V. Lakshmibai, V.B. Mehta and A.J. Parameswaran, Frobenius splittings and blowups, J. Algebra, 208 (1998), 101–128.