Around answer 209140

Definitions

In[1]:=
```
pterm[m_, x_, j_, i_] := Binomial[x + j, j] Binomial[x - 1, j] Binomial[j, i]
  Binomial[m, i] Binomial[i, m - j] 3 / ((2 i - 1) (2 j + 1) (2 m - 2 i - 1))
```

In[2]:=
```
p[m_, x_] := Sum[pterm[m, x, i, j], {i, 0, m}, {j, 0, m}]
```

In[3]:=
```
term[m_, k_, i_, j_] = 3 (-1) ^ (k + j) Binomial[2 k, k] Binomial[j, i]
  Binomial[m, i] Binomial[i, m - j] / (2 (2 i - 1) (2 j + 1) (2 m - 2 i - 1))
```

Out[3]= $\frac{3\ (-1)^{j+k}\ \text{Binomial}[i, -j+m]\ \text{Binomial}[j, i]\ \text{Binomial}[2k, k]\ \text{Binomial}[m, i]}{2\ (-1+2i)\ (1+2j)\ (-1-2i+2m)}$

In[4]:=
```
b[k_, x_] = Binomial[x + k, 2 k] + Binomial[-x + k, 2 k]
```

Out[4]= Binomial[k - x, 2 k] + Binomial[k + x, 2 k]

In[5]:=
```
iterm[m_, k_, i_] =
 (3 Binomial[i, -k + m] Binomial[k, i] Binomial[2 k, k] Binomial[m, i]) /
  ((-1 + 2 i) (-1 - 2 i + 2 m))
```

Out[5]= $\frac{3\ \text{Binomial}[i, -k+m]\ \text{Binomial}[k, i]\ \text{Binomial}[2k, k]\ \text{Binomial}[m, i]}{(-1+2i)\ (-1-2i+2m)}$

In[6]:=
```
g[m_, k_, i_] =
 (3 × 2^(3+2 k) m (1 - 2 i + m) Gamma[3/2 + k] Binomial[k + 1, i - 1] Binomial[m - 1, k + 1]
    Binomial[k + 1, m - i]) / (Gamma[1 / 2] (1 + k) ! )
```

Out[6]= $\frac{1}{\sqrt{\pi}\ (1+k)!} 3 \times 2^{3+2k}\ m\ (1-2i+m)\ \text{Binomial}[1+k, -1+i]$
$\text{Binomial}[1+k, -i+m]\ \text{Binomial}[-1+m, 1+k]\ \text{Gamma}\left[\frac{3}{2}+k\right]$

In[7]:=
```
rel1[m_, k_, d_] := -32 (3 - 2 k)^2 (1 - k + m) (2 - k + m) d[m, -2 + k] +
  4 (1 - k + m) ((-3 + 2 k) (9 + 8 (-2 + k) k) - 2 (-1 + k) (-9 + 8 k) m + 2 k m^2) d[m, -1 + k] +
  k (2 - 2 k + m) (3 - 2 k + m) (1 - 2 k + 2 m) d[m, k]
```

Copy from paper and compare

In[8]:=
```
rel1[m, k, d] == -32 (3 - 2 k) ^2 (-k + m + 1) (-k + m + 2) d[m, k - 2] +
  4 (-k + m + 1) (2 k m^2 - 2 (k - 1) (8 k - 9) m + (2 k - 3) (8 (k - 2) k + 9)) d[m, k - 1] +
  k (-2 k + m + 2) (-2 k + m + 3) (-2 k + 2 m + 1) d[m, k]
```

Out[8]= True

In[9]:=
```
rel2[m_, k_, d_] := -4 (-1 + (-1 + m)^2) d[-1 + m, -1 + k] -
  4 (1 + 2 (-1 + k) + m) (1 - k + m) d[m, -1 + k] + k (-2 + 2 k - m) d[m, k]
```

Copy from paper and compare

```
In[10]:= rel2[m, k, d] == -4 ((m - 1) ^2 - 1) d[m - 1, k - 1] -
           4 (2 (k - 1) + m + 1) (-k + m + 1) d[m, k - 1] + k (2 k - m - 2) d[m, k]

Out[10]= True

In[11]:= longrel[m_, k_, d_] =
          -64 (1 + 2 k)^2 (3 + 2 k) (k - m) (k - m + 1) d[m, k] - 8 (3 + 2 k) (57 + 24 k^4 - 95 m + 42 m^2 - 4 m^3 -
               12 k^3 (-9 + 4 m) + 2 k^2 (99 - 83 m + 13 m^2) + k (171 - 210 m + 64 m^2 - 2 m^3)) d[m, 1 + k] -
            (2 + k) 2 (48 k^4 + k^3 (312 - 96 m) + 8 k^2 (99 - 59 m + 7 m^2) +
               k (924 - 808 m + 192 m^2 - 8 m^3) - 9 (-46 + 53 m - 19 m^2 + 2 m^3)) d[m, 2 + k] -
            (2 + k) (3 + k) (3 + 2 k - m) (4 + 2 k - m) (5 - 2 (-k + m)) d[m, 3 + k]

Out[11]= -64 (1 + 2 k)^2 (3 + 2 k) (k - m) (1 + k - m) d[m, k] -
          8 (3 + 2 k) (57 + 24 k^4 - 95 m + 42 m^2 - 4 m^3 - 12 k^3 (-9 + 4 m) +
              2 k^2 (99 - 83 m + 13 m^2) + k (171 - 210 m + 64 m^2 - 2 m^3)) d[m, 1 + k] -
          2 (2 + k) (48 k^4 + k^3 (312 - 96 m) + 8 k^2 (99 - 59 m + 7 m^2) +
              k (924 - 808 m + 192 m^2 - 8 m^3) - 9 (-46 + 53 m - 19 m^2 + 2 m^3)) d[m, 2 + k] -
          (2 + k) (3 + k) (3 + 2 k - m) (4 + 2 k - m) (5 - 2 (-k + m)) d[m, 3 + k]

In[12]:= int[m_, k_, d_] = 2 (2 k + 1) d[m, k] + (k + 1) d[m, k + 1]

Out[12]= 2 (1 + 2 k) d[m, k] + (1 + k) d[m, 1 + k]

In[13]:= frac1[m_, k_, i_] =
          FullSimplify[((6 (-1)^(2 k) ((-1 + 2 k - 2 m) (-1 + m) m) Binomial[i, 1 - k + m]
                Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k] Binomial[m, i]) /
              ((-1 + 2 i) (2 + 4 i - 4 m))), Element[k, Integers] && k >= 0]

                         1
Out[13]= - ------------------------- 3 (-1 + m) m (1 - 2 k + 2 m) Binomial[i, 1 - k + m]
           (-1 + 2 i) (1 + 2 i - 2 m)
           Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k] Binomial[m, i]

In[14]:= frac2[m_, k_, i_] =
          FullSimplify[6 (-1)^(2 k) (-1 + k - m) Binomial[i, 1 - k + m] Binomial[-1 + k, i]
             Binomial[2 (-1 + k), -1 + k] Binomial[m, i], Element[k, Integers] && k >= 0]

Out[14]= 6 (-1 + k - m) Binomial[i, 1 - k + m]
          Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k] Binomial[m, i]

In[15]:= catalan[i_] = Binomial[2 i, i] / (i + 1)

         Binomial[2 i, i]
Out[15]= ----------------
              1 + i

In[16]:= atjumps::usage :=
          "If the elements of oldlist are of a favorable form, atjumps[oldlist,i]
            finds integers around the jumps of the Floor as a function of i."

In[17]:= atjumps[oldlist_, ii_] :=
          Module[{newlist, tmp, coefs = Abs[Coefficient[oldlist, ii]]},
           tmp = Flatten[Table[Table[ii /. Solve[oldlist[[t]] == j (2 n + 1), ii],
                {j, 0, coefs[[t]] - 1}], {t, Length[oldlist]}]];
           (* Print["jumps in ",oldlist," at ",tmp]; *)
           newlist = FullSimplify[Floor[ii /. ii → tmp], Element[m | n | i, Integers]];
           If[newlist == {}, newlist = {0}];
           Union[newlist - 1, newlist, newlist + 1]]
```

```
In[18]:= ncheck[{numlist_, denomlist_}, bound_] :=
          FullSimplify[Sum[Floor[numlist[[j]] / (2 n + 1)], {j, Length[numlist]}] -
            Sum[Floor[denomlist[[j]] / (2 n + 1)], {j, Length[denomlist]}],
           Element[n, Integers] && n ≥ bound]

In[19]:= numlist = {(2 i - 2), (2 m), (-2 i + 2 m - 2)}

Out[19]= {-2 + 2 i, 2 m, -2 - 2 i + 2 m}

In[20]:= denomlist = {i, i, (2 i - 1), ((m - i)), ((m - i)), (-1 - 2 i + 2 m)}

Out[20]= {i, i, -1 + 2 i, -i + m, -i + m, -1 - 2 i + 2 m}

In[21]:= numlist2 = {i, (-2 + 2 k), m, (1 - 2 k + 2 m), (-2 - 2 i + 2 m)}

Out[21]= {i, -2 + 2 k, m, 1 - 2 k + 2 m, -2 - 2 i + 2 m}

In[22]:= denomlist2 = {(2 i), (-1 + k), (-1 - i + k),
            (-1 - 2 i + 2 m), (-2 k + 2 m), (-1 + i + k - m), (-i + m), (1 - k + m)}

Out[22]= {2 i, -1 + k, -1 - i + k, -1 - 2 i + 2 m, -2 k + 2 m, -1 + i + k - m, -i + m, 1 - k + m}

In[23]:= numlist3 = {(2 k - 2), (m - 2)}

Out[23]= {-2 + 2 k, -2 + m}

In[24]:= denomlist3 = {i, (k - 1), (-i + k - 1), (m - i), (m - k), (i + k - m - 1)}

Out[24]= {i, -1 + k, -1 - i + k, -i + m, -k + m, -1 + i + k - m}

In[25]:= frac[numlist_, denomlist_] := Product[numlist[[j]] !, {j, Length[numlist]}] /
           Product[denomlist[[j]] !, {j, Length[denomlist]}]

In[26]:= test[numlist_, denomlist_, q_] :=
          Sum[Floor[numlist[[j]] / q], {j, Length[numlist]}] -
           Sum[Floor[denomlist[[j]] / q], {j, Length[denomlist]}]

In[27]:= flow::usage := "flow[m,k] is d[m,k] evaluated by recursions"

In[28]:= Clear[flow]

In[29]:= flow[0, 0] = 3 / 2

Out[29]= 3/2

In[30]:= flow[1, 0] = 1

Out[30]= 1

In[31]:= flow[1, 1] = -2

Out[31]= -2

In[32]:= flow[x_Integer, y_Integer] := 0 /; x < 2

In[33]:= flow[x_Integer, y_Integer] := 0 /; y < 2

In[34]:= flow[0, y_Integer] = 0

Out[34]= 0
```

In[35]:=
```
flow[1, y_Integer] = 0
```
Out[35]= 0

In[36]:=
```
flow[2, y_Integer] := If[y == 2, 6, 0]
```

In[37]:=
```
flow[x_Integer, y_Integer] := 0 /; (x > 2 y - 2)
```

In[38]:=
```
flow[x_Integer, y_Integer] :=
 (flow[x, y] = (-4 (-1 + (-1 + x)^2) flow[-1 + x, -1 + y] - 4 (1 + x + 2 (-1 + y))
      (1 + x - y) flow[x, -1 + y]) / (y (2 + x - 2 y))) /; (x < 2 y - 2)
```

In[39]:=
```
flow[x_, y_] :=
 ((-2 + x) (8 (1 - 2 y)^2 (-x + y) flow[-1 + x, -1 + y] + (1 + x - 2 y) (-1 + 2 x - 2 y)
      (-1 + x + 2 y) flow[-1 + x, y])) / ((1 - x) x (1 + 2 x) (x - y)) /; (x == 2 y - 2)
```

Tables

In[40]:=
```
Table[p[m, x], {m, 0, 12}, {x, -6, 6}] // MatrixForm
```
Out[40]//MatrixForm=

3	3	3	3	3	3	3	3	3	3	3	3	3
-70	-48	-30	-16	-6	0	2	0	-6	-16	-30	-48	-70
630	300	120	36	6	0	0	0	6	36	120	300	630
2688	840	192	24	0	0	0	0	0	24	192	840	2688
6820	1320	150	4	0	0	0	0	0	4	150	1320	6820
11 592	1296	60	0	0	0	0	0	0	0	60	1296	11 592
14 112	840	12	0	0	0	0	0	0	0	12	840	14 112
12 320	336	0	0	0	0	0	0	0	0	0	336	12 320
7236	60	0	0	0	0	0	0	0	0	0	60	7236
2520	0	0	0	0	0	0	0	0	0	0	0	2520
392	0	0	0	0	0	0	0	0	0	0	0	392
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

In[41]:=
```
Table[b[k, x], {k, 0, 8}, {x, -6, 6}] // MatrixForm
```
Out[41]//MatrixForm=

2	2	2	2	2	2	2	2	2	2	2	2	2
36	25	16	9	4	1	0	1	4	9	16	25	36
105	50	20	6	1	0	0	0	1	6	20	50	105
112	35	8	1	0	0	0	0	0	1	8	35	112
54	10	1	0	0	0	0	0	0	0	1	10	54
12	1	0	0	0	0	0	0	0	0	0	1	12
1	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

In[42]:=
```
Table[Sum[term[m, k, i, j], {i, 0, m}, {j, k, m}],
  {m, 0, 10}, {k, 0, 10}] // MatrixForm
```
Out[42]//MatrixForm=

$\frac{3}{2}$	0	0	0	0	0	0	0	0	0	0
1	-2	0	0	0	0	0	0	0	0	0
0	0	6	0	0	0	0	0	0	0	0
0	0	0	24	0	0	0	0	0	0	0
0	0	0	4	118	0	0	0	0	0	0
0	0	0	0	60	696	0	0	0	0	0
0	0	0	0	12	720	4824	0	0	0	0
0	0	0	0	0	336	8288	38 240	0	0	0
0	0	0	0	0	60	6516	95 928	336 822	0	0
0	0	0	0	0	0	2520	109 872	1 131 732	3 215 544	0
0	0	0	0	0	0	392	67 904	1 735 320	13 647 840	32 651 544

Checks

```
In[43]:= Table[p[m, k], {k, 10}, {m, 2 k - 1, 23}] // Flatten // Union

Out[43]= {0}

In[44]:= Table[Binomial[i, j], {i, 0, 10}, {j, i + 1, 10}] // Flatten // Union

Out[44]= {0}

In[45]:= Table[p[m, x] == Sum[flow[m, k] b[k, x], {k, 0, m}],
           {m, -2, 14}, {x, -4, 20}] // Flatten // Union

Out[45]= {True}

In[46]:= Table[flow[m, k] == Sum[term[m, k, i, j], {i, 0, m}, {j, k, m}],
           {m, 0, 10}, {k, 0, 10}] // Flatten // Union

Out[46]= {True}

In[47]:= FullSimplify[(-1) ^j Binomial[x + j, j] Binomial[x - 1, j] -
           (-1) ^ (j - 1) Binomial[x + j - 1, j - 1] Binomial[x - 1, j - 1] ==
          (-1) ^j Binomial[2 j, j] b[j, x] / 2, Element[j | x, Integers]]

Out[47]= True

In[48]:= (-1) ^j Binomial[x + j, j] Binomial[x - 1, j] -
           (-1) ^ (j - 1) Binomial[x + j - 1, j - 1] Binomial[x - 1, j - 1] ==
          (-1) ^j Binomial[2 j, j] b[j, x] / 2 /. j → 0

Out[48]= True

In[49]:= term[m, k, i, j]
```

Out[49]=
$$\frac{3\ (-1)^{j+k}\ \text{Binomial}[i, -j+m]\ \text{Binomial}[j, i]\ \text{Binomial}[2\,k, k]\ \text{Binomial}[m, i]}{2\ (-1+2\,i)\ (1+2\,j)\ (-1-2\,i+2\,m)}$$

```
In[50]:= Table[Sum[term[m, k, i, j], {i, 0, m}, {j, k, m}],
           {k, 2, 10}, {m, 2 k - 2 + 1, 30}] // Flatten // Union

Out[50]= {0}

In[51]:= rel2[m + 1, k + 1, d]
```

Out[51]=
$$-4\ (-1+m^2)\ d[m, k] - 4\ (1-k+m)\ (2+2\,k+m)\ d[1+m, k] + (1+k)\ (-3+2\ (1+k)-m)\ d[1+m, 1+k]$$

```
In[52]:= 2 (2 k + 1) term[m, k, i, j] + (k + 1) term[m, k + 1, i, j]
```

Out[52]=
$$\frac{3\ (-1)^{j+k}\ (1+2\,k)\ \text{Binomial}[i, -j+m]\ \text{Binomial}[j, i]\ \text{Binomial}[2\,k, k]\ \text{Binomial}[m, i]}{(-1+2\,i)\ (1+2\,j)\ (-1-2\,i+2\,m)} +$$
$$\left(3\ (-1)^{1+j+k}\ (1+k)\ \text{Binomial}[i, -j+m]\ \text{Binomial}[j, i]\ \text{Binomial}[2\ (1+k), 1+k]\ \text{Binomial}[m, i]\right) \Big/ \left(2\ (-1+2\,i)\ (1+2\,j)\ (-1-2\,i+2\,m)\right)$$

```
In[53]:= % // FullSimplify

Out[53]= 0
```

In[54]:=
```
%% /. k → 0
```

Out[54]=

$$\frac{3\,(-1)^{j}\,\text{Binomial}[i,-j+m]\,\text{Binomial}[j,i]\,\text{Binomial}[m,i]}{(-1+2\,i)\,(1+2\,j)\,(-1-2\,i+2\,m)}+\frac{3\,(-1)^{1+j}\,\text{Binomial}[i,-j+m]\,\text{Binomial}[j,i]\,\text{Binomial}[m,i]}{(-1+2\,i)\,(1+2\,j)\,(-1-2\,i+2\,m)}$$

In[55]:=
```
% // FullSimplify
```

Out[55]= 0

Is valid for k > 0. Also for k = -1, k < -1, k = 0.

In[56]:=
```
Table[2 (2 k + 1) flow[m, k] + (k + 1) flow[m, k + 1] - Sum[iterm[m, k, i], {i, 0, m}],
    {m, 0, 44}, {k, 0, m}] // Flatten // Union
```

Out[56]= {0}

In[57]:=
```
Table[Plus @@ {-32 (1 + 2 k) (3 + 2 k) (k - m) (1 + k - m) iterm[m, k, i] , -4 (1 + k - m)
      (57 + 110 k + 72 k^2 + 16 k^3 - 34 m - 46 k m - 16 k^2 m + 4 m^2 + 2 k m^2) iterm[m, k + 1, i] ,
    -(2 + k) (5 + 2 k - 2 m) (3 + 2 k - m) (4 + 2 k - m) iterm[m, k + 2, i], -g[m, k, i + 1],
    +g[m, k, i]}, {m, 0, 55}, {k, 0, m}, {i, 0, m}] // Flatten // Union
```

Out[57]= {0}

In[58]:=
```
Plus @@ {-32 (1 + 2 k) (3 + 2 k) (k - m) (1 + k - m) iterm[m, k, i] ,
   -4 (1 + k - m) (57 + 110 k + 72 k^2 + 16 k^3 - 34 m - 46 k m - 16 k^2 m + 4 m^2 + 2 k m^2)
    iterm[m, k + 1, i] , -(2 + k) (5 + 2 k - 2 m) (3 + 2 k - m)
    (4 + 2 k - m) iterm[m, k + 2, i], -g[m, k, i + 1], +g[m, k, i]}
```

Out[58]=

$$-\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}96\,(1+2\,k)\,(3+2\,k)\,(k-m)\,(1+k-m)\,\text{Binomial}[i,-k+m]\,\text{Binomial}[k,i]\,\text{Binomial}[2\,k,k]\,\text{Binomial}[m,i]-\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}12\,(1+k-m)\,\left(57+110\,k+72\,k^2+16\,k^3-34\,m-46\,k\,m-16\,k^2\,m+4\,m^2+2\,k\,m^2\right)\text{Binomial}[i,-1-k+m]\,\text{Binomial}[1+k,i]\,\text{Binomial}[2\,(1+k),1+k]\,\text{Binomial}[m,i]+\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}3\,(-2-k)\,(5+2\,k-2\,m)\,(3+2\,k-m)\,(4+2\,k-m)\,\text{Binomial}[i,-2-k+m]\,\text{Binomial}[2+k,i]\,\text{Binomial}[2\,(2+k),2+k]\,\text{Binomial}[m,i]-\frac{1}{\sqrt{\pi}\,(1+k)!}3\times 2^{3+2\,k}\,m\,\left(1-2\,(1+i)+m\right)\text{Binomial}[1+k,i]\,\text{Binomial}[1+k,-1-i+m]\,\text{Binomial}[-1+m,1+k]\,\text{Gamma}\left[\frac{3}{2}+k\right]+\frac{1}{\sqrt{\pi}\,(1+k)!}3\times 2^{3+2\,k}\,m\,(1-2\,i+m)\,\text{Binomial}[1+k,-1+i]\,\text{Binomial}[1+k,-i+m]\,\text{Binomial}[-1+m,1+k]\,\text{Gamma}\left[\frac{3}{2}+k\right]$$

In[59]:=
```
% // FullSimplify
```

Out[59]= 0

Is valid if i >= 0, k >= 0, m >= 1. Also, if i < 0 you are left with nothing. So we have now case k >= 0, m >= 1.
If m < 0 then iterm vanishes by definition.

In[60]:=
```
Binomial[i, -k + m] Binomial[k, i] Binomial[2 k, k] Binomial[m, i] -
 Binomial[i, -1 - k + m] Binomial[1 + k, i] Binomial[2 (1 + k), 1 + k] Binomial[m, i] +
 Binomial[i, -2 - k + m] Binomial[2 + k, i] Binomial[2 (2 + k), 2 + k] Binomial[m, i] -
 Binomial[1 + k, i] Binomial[1 + k, -1 - i + m] Binomial[-1 + m, 1 + k] +
 Binomial[1 + k, -1 + i] Binomial[1 + k, -i + m] Binomial[-1 + m, 1 + k]
```

Out[60]=
```
-Binomial[1 + k, i] Binomial[1 + k, -1 - i + m] Binomial[-1 + m, 1 + k] +
 Binomial[1 + k, -1 + i] Binomial[1 + k, -i + m] Binomial[-1 + m, 1 + k] +
 Binomial[i, -k + m] Binomial[k, i] Binomial[2 k, k] Binomial[m, i] -
 Binomial[i, -1 - k + m] Binomial[1 + k, i] Binomial[2 (1 + k), 1 + k] Binomial[m, i] +
 Binomial[i, -2 - k + m] Binomial[2 + k, i] Binomial[2 (2 + k), 2 + k] Binomial[m, i]
```

In[61]:=
$$-\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}\,96\,(1+2\,k)\,(3+2\,k)\,(k-m)\,(1+k-m)\,\text{Binomial}[i,-k+m]\,\text{Binomial}[k,i]\,\text{Binomial}[2\,k,k]\,\text{Binomial}[m,i]-\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}\,12\,(1+k-m)\,\left(57+110\,k+72\,k^2+16\,k^3-34\,m-46\,k\,m-16\,k^2\,m+4\,m^2+2\,k\,m^2\right)\,\text{Binomial}[i,-1-k+m]\,\text{Binomial}[1+k,i]\,\text{Binomial}[2\,(1+k),1+k]\,\text{Binomial}[m,i]+\frac{1}{(-1+2\,i)\,(-1-2\,i+2\,m)}\,3\,(-2-k)\,(5+2\,k-2\,m)\,(3+2\,k-m)\,(4+2\,k-m)\,\text{Binomial}[i,-2-k+m]\,\text{Binomial}[2+k,i]\,\text{Binomial}[2\,(2+k),2+k]\,\text{Binomial}[m,i]-\frac{1}{\sqrt{\pi}\,(1+k)!}\,3\times 2^{3+2\,k}\,m\,(1-2\,(1+i)+m)\,\text{Binomial}[1+k,i]\,\text{Binomial}[1+k,-1-i+m]\,\text{Binomial}[-1+m,1+k]\,\text{Gamma}\left[\frac{3}{2}+k\right]+\frac{1}{\sqrt{\pi}\,(1+k)!}\,3\times 2^{3+2\,k}\,m\,(1-2\,i+m)\,\text{Binomial}[1+k,-1+i]\,\text{Binomial}[1+k,-i+m]\,\text{Binomial}[-1+m,1+k]\,\text{Gamma}\left[\frac{3}{2}+k\right]\;/.\;m\to 0$$

Out[61]=
$$-\frac{1}{(-1-2\,i)\,(-1+2\,i)}\,96\,k\,(1+k)\,(1+2\,k)\,(3+2\,k)\,\text{Binomial}[0,i]\,\text{Binomial}[i,-k]\,\text{Binomial}[k,i]\,\text{Binomial}[2\,k,k]-\frac{1}{(-1-2\,i)\,(-1+2\,i)}\,12\,(1+k)\,\left(57+110\,k+72\,k^2+16\,k^3\right)\,\text{Binomial}[0,i]\,\text{Binomial}[i,-1-k]\,\text{Binomial}[1+k,i]\,\text{Binomial}[2\,(1+k),1+k]+\frac{1}{(-1-2\,i)\,(-1+2\,i)}\,3\,(-2-k)\,(3+2\,k)\,(4+2\,k)\,(5+2\,k)\,\text{Binomial}[0,i]\,\text{Binomial}[i,-2-k]\,\text{Binomial}[2+k,i]\,\text{Binomial}[2\,(2+k),2+k]$$

In[62]:=
```
% // FullSimplify
```

Out[62]=
$$\frac{12\,i\,\text{Gamma}[4+2\,k]\,\text{Sin}[i\,\pi]\,\text{Sin}[k\,\pi]}{\pi^2\,\text{Gamma}[3-i+k]\,\text{Gamma}[3+i+k]}$$

So we also have case m = 0, i >= 0, k>= 0.

In[63]:=
```
FullSimplify[(i ≥ 0 && k ≥ 0 && m ≥ 1) || (i < 0) || (m == 0 && i ≥ 0 && k ≥ 0),
 Element[k | i | m, Integers]]
```

Out[63]=
```
(k ≥ 0 && m ≥ 0) || i < 0
```

```
In[64]:= Plus @@ {-32 (1 + 2 k) (3 + 2 k) (k - m) (1 + k - m) iterm[m, k] , -4 (1 + k - m)
        (57 + 110 k + 72 k^2 + 16 k^3 - 34 m - 46 k m - 16 k^2 m + 4 m^2 + 2 k m^2) iterm[m, k + 1] ,
      - (2 + k) (5 + 2 k - 2 m) (3 + 2 k - m) (4 + 2 k - m) iterm[m, k + 2]} /.
   iterm[m_, k_] → 2 (2 k + 1) d[m, k] + (k + 1) d[m, k + 1]

Out[64]= -32 (1 + 2 k) (3 + 2 k) (k - m) (1 + k - m) (2 (1 + 2 k) d[m, k] + (1 + k) d[m, 1 + k]) -
   4 (1 + k - m) (57 + 110 k + 72 k^2 + 16 k^3 - 34 m - 46 k m - 16 k^2 m + 4 m^2 + 2 k m^2)
    (2 (1 + 2 (1 + k)) d[m, 1 + k] + (2 + k) d[m, 2 + k]) + (-2 - k) (5 + 2 k - 2 m)
    (3 + 2 k - m) (4 + 2 k - m) (2 (1 + 2 (2 + k)) d[m, 2 + k] + (3 + k) d[m, 3 + k])

In[65]:= % == 2 (3 + 2 k) rel1[m, k + 2, d] + (2 + k) rel1[m, k + 3, d] // FullSimplify

Out[65]= True

In[66]:= Table[pterm[m, k, i, j], {k, 10}, {m, 2 k - 1, 20}, {i, 0, m}, {j, 0, m}] // Flatten //
   Union

Out[66]= {0}

In[67]:= Table[Sum[d[m, k] b[k, 1], {k, 0, m}] == 2 d[m, 0] + d[m, 1], {m, 10}] // MatrixForm

Out[67]//MatrixForm=
   True
   True
   True
   True
   True
   True
   True
   True
   True
   True

In[68]:= Table[Sum[d[m, k] b[k, 2], {k, 0, m}] == 2 d[m, 0] + 4 d[m, 1] + d[m, 2], {m, 10}] //
   MatrixForm

Out[68]//MatrixForm=
   2 d[1, 0] + 4 d[1, 1] == 2 d[1, 0] + 4 d[1, 1] + d[1, 2]
                        True
                        True
                        True
                        True
                        True
                        True
                        True
                        True
                        True

In[69]:= {2 d[m, 0] + d[m, 1], 2 d[m, 0] + 4 d[m, 1] + d[m, 2]} /.
    d[m, 2] → -2 d[m, 0] - 4 d[m, 1] /. d[m, 1] -> -2 d[m, 0]

Out[69]= {0, 0}

In[70]:= rel1[m, 2, d] /. d[m, 2] → -2 d[m, 0] - 4 d[m, 1] /. d[m, 1] -> -2 d[m, 0] //
   FullSimplify

Out[70]= 4 m (1 + m - 2 m^2) d[m, 0]

In[71]:= Factor[4 m (1 + m - 2 m^2)]

Out[71]= -4 (-1 + m) m (1 + 2 m)

In[72]:= rel1[m, 2, d] /. d[m, 2] → 6 d[m, 0] /. d[m, 1] -> -2 d[m, 0] // FullSimplify

Out[72]= 4 m (1 + m - 2 m^2) d[m, 0]
```

```
In[73]:= % == 4 m (m - 1) (-2 m - 1) d[m, 0]
Out[73]= 4 m (1 + m - 2 m^2) d[m, 0] == 4 (-1 - 2 m) (-1 + m) m d[m, 0]

In[74]:= % // FullSimplify
Out[74]= True

In[75]:= FullSimplify[(m > 2 k - 2 || m == 0 || m == 1) &&
           Not[(m > 2 k - 2 ≥ 0) || (m == 2 && k == 0) || (k == 0 && m ≥ 3) ||
             m < 0 || k < 0 || m == 0 || m == 1], Element[k | i | m, Integers]]
Out[75]= False
```

So no case forgotten.

```
In[76]:= Table[rel1[m, k, flow], {k, -10, 3}, {m, 0, 10}] // Flatten // Union
Out[76]= {0}

In[77]:= Table[flow[2 k - 2, k] == p[2 k - 2, k] == pterm[2 k - 2, k, k - 1, k - 1], {k, 2, 10}] //
           Flatten // Union
Out[77]= {True}

In[78]:= Table[flow[2 k - 3, k] == p[2 k - 3, k] == pterm[2 k - 3, k, k - 1, k - 1] +
             pterm[2 k - 3, k, k - 1, k - 1], {k, 3, 10}] // Flatten // Union
Out[78]= {True}

In[79]:= Table[flow[2 k - 2, j], {k, 3, 10}, {j, 0, k - 1}] // Flatten // Union
Out[79]= {0}

In[80]:= Table[flow[2 k - 3, j], {k, 3, 10}, {j, 0, k - 1}] // Flatten // Union
Out[80]= {0}

In[81]:= rel2[m, k, d]
Out[81]= -4 (-1 + (-1 + m)^2) d[-1 + m, -1 + k] -
          4 (1 + 2 (-1 + k) + m) (1 - k + m) d[m, -1 + k] + k (-2 + 2 k - m) d[m, k]

In[82]:= Table[rel2[m, k, flow], {k, -4, 4}, {m, -4, 12}] // Flatten // Union
Out[82]= {0}

In[83]:= Clear[dknown]

In[84]:= dknown[m_, k_] := 0 /; (FullSimplify[(m > 2 k - 2)])

In[85]:= dknown[m_, k_] := pterm[2 k - 2, k, k - 1, k - 1] /; (FullSimplify[(m == 2 k - 2)])

In[86]:= dknown[m_, k_] := pterm[2 k - 3, k, k - 1, k - 1] + pterm[2 k - 3, k, k - 1, k - 1] /;
           (FullSimplify[(m == 2 k - 3)])

In[87]:= rel2[2 k - 3, k, dknown]
Out[87]= -((12 (-1 + (-4 + 2 k)^2) Binomial[-2 + 2 (-1 + k), -2 + k]
              Binomial[-2 + k, 2 (-1 + k) - k] Binomial[-3 + 2 k, -2 + k]) /
            ((-1 + 2 (-2 + 2 (-1 + k)) - 2 (-2 + k)) (-1 + 2 (-2 + k)) (1 + 2 (-2 + k)))) +
          (6 (-1 + k) k Binomial[-3 + 2 k, -1 + k] Binomial[-1 + 2 k, -1 + k]) /
          ((-1 + 2 (-1 + k)) (1 + 2 (-1 + k)) (-1 - 2 (-1 + k) + 2 (-3 + 2 k)))
```

```
In[88]:= FullSimplify[%, Element[k | i | m, Integers] && k ≥ 4]

Out[88]= 0
```

This is valid for k not too small.

```
In[89]:= Table[rel2[2 k - 3, k, flow], {k, -3, 20}] // Flatten // Union

Out[89]= {0}

In[90]:= rel2[2 k - 2, k, dknown]

Out[90]= 0

In[91]:= Table[rel2[2 k - 2, k, flow], {k, -3, 20}] // Flatten // Union

Out[91]= {0}

In[92]:= Table[rel2[2 k - 1, k, dknown], {k, 4, 17}] // Flatten // Union

Out[92]= {0}

In[93]:= Table[rel2[2 k - 1, k, flow], {k, -3, 20}] // Flatten // Union

Out[93]= {0}

In[94]:= Table[rel2[2 k, k, dknown], {k, 4, 17}] // Flatten // Union

Out[94]= {0}

In[95]:= Table[rel2[2 k, k, flow], {k, -3, 20}] // Flatten // Union

Out[95]= {0}

In[96]:= (-7 + 2 k) rel2[2 k - 4, k, d] - rel1[2 k - 4, k, d] // FullSimplify

Out[96]= 16 (-3 + k) (-2 + k) ((7 - 2 k) d[-5 + 2 k, -1 + k] +
            2 (3 - 2 k)^2 d[-4 + 2 k, -2 + k] + (-2 + k) (-5 + 2 k) d[-4 + 2 k, -1 + k])

In[97]:= % /. d → dknown

Out[97]= 16 (-3 + k) (-2 + k)
          ((6 (7 - 2 k) Binomial[-3 + 2 (-1 + k), -2 + k] Binomial[-2 + k, -1 + 2 (-1 + k) - k]
               Binomial[-3 + 2 k, -2 + k]) /
             ((-1 + 2 (-3 + 2 (-1 + k)) - 2 (-2 + k)) (-1 + 2 (-2 + k)) (1 + 2 (-2 + k))) +
            (3 (-2 + k) (-5 + 2 k) Binomial[-2 + 2 (-1 + k), -2 + k]
               Binomial[-2 + k, 2 (-1 + k) - k] Binomial[-3 + 2 k, -2 + k]) /
             ((-1 + 2 (-2 + 2 (-1 + k)) - 2 (-2 + k)) (-1 + 2 (-2 + k)) (1 + 2 (-2 + k))))

In[98]:= % // FullSimplify

Out[98]= 0

In[99]:= Table[rel2[m, k, flow], {k, -4, 2}, {m, -4, 8}] // Flatten // Union

Out[99]= {0}

In[100]:= rel2[m, k, d]

Out[100]= -4 (-1 + (-1 + m)^2) d[-1 + m, -1 + k] -
           4 (1 + 2 (-1 + k) + m) (1 - k + m) d[m, -1 + k] + k (-2 + 2 k - m) d[m, k]
```

```
In[101]:= rel1[m, k, d]

Out[101]= -32 (3 - 2 k)^2 (1 - k + m) (2 - k + m) d[m, -2 + k] +
  4 (1 - k + m) ((-3 + 2 k) (9 + 8 (-2 + k) k) - 2 (-1 + k) (-9 + 8 k) m + 2 k m^2) d[m, -1 + k] +
  k (2 - 2 k + m) (3 - 2 k + m) (1 - 2 k + 2 m) d[m, k]

In[102]:= (-1 + k) (-1 + 2 k - 2 m) (-3 + 2 k - m) (4 - 2 k + m) rel2[m, k, d] +
   4 (-1 + (-1 + m) ^2) rel1[m - 1, k - 1, d] -
   32 (5 - 2 k) ^2 (-2 + k - m) (-1 + k - m) rel2[m, k - 2, d] +
   4 (1 - k + m) (-99 + 16 k^3 - 2 m (32 + m) - 8 k^2 (11 + 2 m) + 2 k (81 + m (31 + m)))
    rel2[m, k - 1, d] - (-1 + k) (-4 + 2 k - m) rel1[m, k, d] -
   4 (-1 + k - m) (-5 + 2 k + m) rel1[m, k - 1, d] // FullSimplify

Out[102]= 0

In[103]:= int[m, k, d]

Out[103]= 2 (1 + 2 k) d[m, k] + (1 + k) d[m, 1 + k]

In[104]:= -rel1[m, k, d] + 2 (m - 1) m (2 m + 1) d[m, k - 1] +
   (2 - 2 k + m) (3 - 2 k + m) (1 - 2 k + 2 m) int[m, k - 1, d] +
   16 (3 - 2 k) (-2 + k - m) (-1 + k - m) int[m, k - 2, d] // FullSimplify

Out[104]= 0

In[105]:= frac1[m, k, i] == 3 (m - 1) m Binomial[2 k - 2, k - 1] (-2 k + 2 m + 1) Binomial[k - 1, i]
    Binomial[m, i] Binomial[i, -k + m + 1] / ((2 i - 1) (2 m - 2 i - 1)) // FullSimplify

Out[105]= True

In[106]:= frac2[m, k, i]

Out[106]= 6 (-1 + k - m) Binomial[i, 1 - k + m]
  Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k] Binomial[m, i]

In[107]:= iterm[m, k - 1, i]

Out[107]= 3 Binomial[i, 1 - k + m] Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k] Binomial[m, i]
          ---------------------------------------------------------------------------------------
                                       (-1 + 2 i) (-1 - 2 i + 2 m)

In[108]:= iterm[m, k - 2, i]

Out[108]= 3 Binomial[i, 2 - k + m] Binomial[-2 + k, i] Binomial[2 (-2 + k), -2 + k] Binomial[m, i]
          ---------------------------------------------------------------------------------------
                                       (-1 + 2 i) (-1 - 2 i + 2 m)

In[109]:= frac1[m, k, i] + frac2[m, k, i] ==
   (2 - 2 k + m) (3 - 2 k + m) (1 - 2 k + 2 m) iterm[m, k - 1, i] +
    16 (3 - 2 k) (-2 + k - m) (-1 + k - m) iterm[m, k - 2, i] // FullSimplify

Out[109]= True
```

Is valid if i >= 0, k >= 2, m >= 0. Also if i < 0.
frac1[m,k,i] is nonzero only if m >= k-1 >= i >= m-k+1 >= 0
frac2[m,k,i] is nonzero only if m >= k-1 >= i >= m-k+1 > 0
iterm[m,k-1,i] is nonzero only if m >= k-1 >= i >= m-k+1 >= 0
iterm[m,k-2,i] is nonzero only if m >= k-2 >= i >= m-k+2 >= 0.
So we may assume m >= k-2, k-1 >= i >= m-k+1, i >= 0, k >= 1, m >= 0.
We still need the case i >= 0, k = 1, m >= 0.

In[110]:= **frac1[m, 1, i] + frac2[m, 1, i]**

Out[110]= $-6\,m\,\text{Binomial}[0, i]\,\text{Binomial}[i, m]\,\text{Binomial}[m, i] - \frac{3\,(-1+m)\,m\,(-1+2\,m)\,\text{Binomial}[0, i]\,\text{Binomial}[i, m]\,\text{Binomial}[m, i]}{(-1+2\,i)\,(1+2\,i-2\,m)}$

In[111]:= **(2 - 2 k + m) (3 - 2 k + m) (1 - 2 k + 2 m) iterm[m, k - 1, i] + 16 (3 - 2 k) (-2 + k - m) (-1 + k - m) iterm[m, k - 2, i] /. k → 1**

Out[111]= $\frac{3\,m\,(1+m)\,(-1+2\,m)\,\text{Binomial}[0, i]\,\text{Binomial}[i, m]\,\text{Binomial}[m, i]}{(-1+2\,i)\,(-1-2\,i+2\,m)}$

In[112]:= **% == %% /. i → 0**

Out[112]= $-3\,m\,(1+m)\,\text{Binomial}[0, m] == -6\,m\,\text{Binomial}[0, m] + \frac{3\,(-1+m)\,m\,(-1+2\,m)\,\text{Binomial}[0, m]}{1-2\,m}$

In[113]:= **% /. m → 0**

Out[113]= True

In[114]:= **frac1[k - 1, k, i] / (6 m (m - 1)) /. m → k - 1 /. i → k - 1 // FullSimplify**

Out[114]= $\frac{\text{Binomial}[2\,(-1+k), -1+k]}{-6+4\,k}$

In[115]:= **% == catalan[k - 2] // FullSimplify**

Out[115]= True

In[116]:= **frac1[k - 1, k, i] / (6 m (m - 1)) /. m → k - 1 /. i → 0 // FullSimplify**

Out[116]= $\frac{\text{Binomial}[2\,(-1+k), -1+k]}{-6+4\,k}$

In[117]:= **% == catalan[k - 2] // FullSimplify**

Out[117]= True

In[118]:= **-frac[numlist, denomlist] / 2**

Out[118]= $-\frac{(-2+2\,i)!\,(2\,m)!\,(-2-2\,i+2\,m)!}{2\,(i!)^2\,(-1+2\,i)!\,((-i+m)!)^2\,(-1-2\,i+2\,m)!}$

In[119]:= **frac1[m, m + 1, i] / (6 m (m - 1))**

Out[119]= $-\frac{(1+2\,m-2\,(1+m))\,\text{Binomial}[m, i]^2\,\text{Binomial}[2\,m, m]}{2\,(-1+2\,i)\,(1+2\,i-2\,m)}$

In[120]:= **% == %% // FullSimplify**

Out[120]= True

In[121]:= **test[numlist, denomlist, 2 n + 1]**

Out[121]= $-2\,\text{Floor}\left[\frac{i}{1+2\,n}\right] + \text{Floor}\left[\frac{-2+2\,i}{1+2\,n}\right] - \text{Floor}\left[\frac{-1+2\,i}{1+2\,n}\right] + \text{Floor}\left[\frac{2\,m}{1+2\,n}\right] - 2\,\text{Floor}\left[\frac{-i+m}{1+2\,n}\right] + \text{Floor}\left[\frac{-2-2\,i+2\,m}{1+2\,n}\right] - \text{Floor}\left[\frac{-1-2\,i+2\,m}{1+2\,n}\right]$

In[122]:=
```
Table[test[numlist, denomlist, 2 n + 1], {n, 9}, {m, 2 n + 1}, {i, 2 n + 1}] // Flatten //
 Union
```
Out[122]= {0, 1, 2, 3}

In[123]:=
```
Length[ivals = atjumps[Union[numlist, denomlist], i]]
```
Out[123]= 15

In[124]:=
```
Table[pairlist2 = {numlist, denomlist} /. i → ivals[[y]];
 mvals = atjumps[Union @@ pairlist2, m];
 Table[ ncheck[pairlist2 /. m -> mvals[[x]], 5], {x, Length[mvals]}],
 {y, Length[ivals]}]
```
Out[124]=
```
{{3, 1, 2, 2, 2, 1, 3}, {1, 0, 0, 0, 0, 0, 0, 1}, {1, 2, 0, 0, 0, 0, 0, 1, 0, 1},
 {1, 2, 2, 0, 0, 0, 0, 0, 1, 1, 0, 1}, {1, 2, 2, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1},
 {1, 2, 0, 0, 0, 0, 0, 1, 0, 1}, {1, 0, 0, 0, 0, 0, 0, 1},
 {3, 1, 2, 2, 2, 1, 3}, {1, 2, 2, 2, 2, 3, 3, 1, 1, 1, 1, 2, 2},
 {1, 2, 2, 2, 2, 3, 1, 1, 1, 2, 1, 2}, {0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1},
 {1, 1, 2, 2, 0, 1, 1, 1, 1, 2}, {1, 1, 2, 2, 0, 1, 1, 1, 1, 2},
 {0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1}, {1, 2, 2, 2, 2, 3, 1, 1, 1, 2, 1, 2}}
```

In[125]:=
```
tmp = % // Flatten; {Length[tmp], Union[tmp]}
```
Out[125]= {153, {0, 1, 2, 3}}

In[126]:=
```
frac1[m, k, i] / (6 m (m - 1) catalan[i - 1]) // FullSimplify
```
Out[126]=
```
- ((i (1 - 2 k + 2 m) Binomial[i, 1 - k + m] Binomial[-1 + k, i] Binomial[2 (-1 + k), -1 + k]
     Binomial[m, i]) / (2 (-1 + 2 i) (1 + 2 i - 2 m) Binomial[2 (-1 + i), -1 + i]))
```

In[127]:=
```
frac[numlist2, denomlist2]
```
Out[127]=
```
(i ! (-2 + 2 k) ! m ! (-2 - 2 i + 2 m) ! (1 - 2 k + 2 m) !) /
 ((2 i) ! (-1 + k) ! (-1 - i + k) ! (-1 + i + k - m) !
   (-i + m) ! (1 - k + m) ! (-1 - 2 i + 2 m) ! (-2 k + 2 m) !)
```

In[128]:=
```
% == %% // FullSimplify
```
Out[128]= True

In[129]:=
```
test[numlist2, denomlist2, 2 n + 1]
```
Out[129]=

$$\text{Floor}\left[\frac{i}{1+2n}\right]-\text{Floor}\left[\frac{2i}{1+2n}\right]-\text{Floor}\left[\frac{-1+k}{1+2n}\right]-\text{Floor}\left[\frac{-1-i+k}{1+2n}\right]+\text{Floor}\left[\frac{-2+2k}{1+2n}\right]-\text{Floor}\left[\frac{-1+i+k-m}{1+2n}\right]+\text{Floor}\left[\frac{m}{1+2n}\right]-\text{Floor}\left[\frac{-i+m}{1+2n}\right]-\text{Floor}\left[\frac{1-k+m}{1+2n}\right]+\text{Floor}\left[\frac{-2-2i+2m}{1+2n}\right]-\text{Floor}\left[\frac{-1-2i+2m}{1+2n}\right]-\text{Floor}\left[\frac{-2k+2m}{1+2n}\right]+\text{Floor}\left[\frac{1-2k+2m}{1+2n}\right]$$

In[130]:=
```
Table[test[numlist2, denomlist2, 2 n + 1],
   {n, 9}, {k, 2 n + 1}, {m, 2 n + 1}, {i, 2 n + 1}] // Flatten // Union
```
Out[130]= {0, 1, 2, 3, 4}

In[131]:=
```
Length[kvals2 = atjumps[Union[numlist2, denomlist2], k]]
```
Out[131]= 20

In[132]:=
```
kvals = atjumps[Union[numlist2, denomlist2], k]
```
Out[132]=
```
{0, 1, 2, i, 1 + i, 2 + i, -1 + m, m, 1 + m, 2 + m, -i + m,
 1 - i + m, 2 - i + m, -2 + m - n, -1 + m - n, m - n, 1 + m - n, n, 1 + n, 2 + n}
```

In[133]:=
```
Table[pair1 = {numlist2, denomlist2} /. k → kvals[[z]];
 mvals = atjumps[Union @@ pair1, m];
 Table[pair2 = pair1 /. m → mvals[[y]];
  ivals = atjumps[Union @@ pair2, i];
  Table[ ncheck[pair2 /. i → ivals[[x]], 6], {x, Length[ivals]}],
  {y, Length[mvals]}], {z, Length[kvals]}]
```

Out[133]=
```
{{{1, 1, 1, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1}, {0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0},
  {1, 1, 2, 2, 2, 2, 2, 1, 2, 1, 1, 1}, {1, 1, 2, 2, 2, 2, 2, 2, 1, 2, 0, 2, 0},
  {1, 1, 2, 1, 2, 2, 2, 2, 1, 2, 1}, {1, 1, 1, 2, 2, 2, 2, 2, 1},
  {1, 0, 2, 2, 2, 3, 1}, {1, 0, 1, 2, 2, 2, 3, 2, 1}, {1, 0, 1, 1, 2, 2, 2, 3, 2, 2, 1},
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```

In[134]:= **tmp = % // Flatten; {Length[tmp], Union[tmp]}**

Out[134]= {3508, {0, 1, 2, 3, 4}}

In[135]:= **frac2[m, k, i] / (6 m (m - 1))**

Out[135]= $\frac{1}{(-1+m)\,m}(-1+k-m)\,\text{Binomial}[i, 1-k+m]\,\text{Binomial}[-1+k, i]\,\text{Binomial}[2\,(-1+k), -1+k]\,\text{Binomial}[m, i]$

In[136]:= **frac[numlist3, denomlist3]**

Out[136]= $\frac{(-2+2k)!\,(-2+m)!}{i!\,(-1+k)!\,(-1-i+k)!\,(-1+i+k-m)!\,(-i+m)!\,(-k+m)!}$

In[137]:= **-% == %% // FullSimplify**

Out[137]= True

In[138]:=
```
test[numlist3, denomlist3, 2 n + 1]
```

Out[138]=

$$-\text{Floor}\left[\frac{i}{1+2n}\right]-\text{Floor}\left[\frac{-1+k}{1+2n}\right]-\text{Floor}\left[\frac{-1-i+k}{1+2n}\right]+\text{Floor}\left[\frac{-2+2k}{1+2n}\right]-\text{Floor}\left[\frac{-1+i+k-m}{1+2n}\right]+\text{Floor}\left[\frac{-2+m}{1+2n}\right]-\text{Floor}\left[\frac{-i+m}{1+2n}\right]-\text{Floor}\left[\frac{-k+m}{1+2n}\right]$$

In[139]:=
```
Table[test[numlist3, denomlist3, 2 n + 1],
    {n, 9}, {k, 2 n + 1}, {m, 2 n + 1}, {i, 2 n + 1}] // Flatten // Union
```

Out[139]= {0, 1, 2, 3, 4}

In[140]:=
```
Length[kvals2 = atjumps[Union[numlist3, denomlist3], k]]
```

Out[140]= 15

In[141]:=
```
kvals = atjumps[Union[numlist3, denomlist3], k]
```

Out[141]= {0, 1, 2, i, 1 + i, 2 + i, -1 + m, m, 1 + m, -i + m, 1 - i + m, 2 - i + m, n, 1 + n, 2 + n}

In[142]:=
```
Table[pair1 = {numlist3, denomlist3} /. k → kvals[[z]];
 mvals = atjumps[Union @@ pair1, m];
 Table[pair2 = pair1 /. m → mvals[[y]];
  ivals = atjumps[Union @@ pair2, i];
  Table[ ncheck[pair2 /. i → ivals[[x]], 6], {x, Length[ivals]}],
  {y, Length[mvals]}], {z, Length[kvals]}]
```

```
Out[142]= {{{2, 2, 2, 2}, {1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1},
    {2, 2, 2, 2, 2, 2, 2}, {2, 2, 2, 2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1, 2, 2},
    {2, 2, 2, 1, 1, 2, 2}, {2, 2, 1, 1, 2, 2}, {2, 1, 1, 2, 2}},
   {{2, 0, 2}, {1, 0, 0, 1}, {2, 1, 2, 1, 2}, {2, 1, 2, 2, 1, 2}, {2, 1, 2, 1, 2, 2},
    {1, 0, 0, 1, 1}, {2, 0, 2, 2}}, {{2, 0, 0, 2}, {2, 1, 0, 1, 2}, {2, 1, 1, 1, 1, 2},
    {2, 1, 1, 2, 2, 2}, {1, 0, 0, 1, 1}, {1, 0, 0, 1}, {2, 1, 1, 2}},
   {{2, 1, 0, 2, 2, 2, 2}, {3, 2, 2, 1, 3, 3, 3, 3, 3}, {3, 2, 2, 1, 1, 3, 3, 3, 4, 3},
    {4, 2, 2, 2, 3, 3, 3, 3, 4}, {2, 1, 0, 1, 1, 1, 1, 2}, {4, 1, 2, 2, 2, 2, 4},
    {2, 1, 2, 1, 1, 1, 1, 2}, {3, 2, 2, 1, 1, 1, 1, 2, 3}, {3, 2, 0, 1, 1, 1, 3, 3}},
   {{1, 0, 0, 1, 1, 1, 1, 1}, {2, 1, 0, 1, 2, 2, 2, 3, 2},
    {2, 1, 1, 0, 1, 2, 2, 2, 3, 2, 2}, {3, 2, 1, 1, 1, 2, 2, 2, 2, 3},
    {2, 0, 0, 1, 1, 1, 1, 2}, {2, 0, 0, 0, 0, 0, 1, 2}, {1, 0, 0, 0, 0, 0, 1},
    {1, 1, 1, 0, 0, 0, 0, 1}, {2, 1, 0, 0, 0, 0, 0, 1, 2, 2}, {3, 2, 0, 1, 1, 1, 3, 3}},
   {{1, 1, 0, 0, 1, 1, 1, 2, 1}, {2, 2, 1, 1, 1, 2, 2, 2, 3, 2, 2},
    {2, 2, 1, 0, 1, 1, 2, 2, 2, 3, 3, 2}, {3, 2, 2, 1, 1, 1, 2, 2, 2, 2, 3},
    {2, 1, 0, 0, 1, 1, 1, 1, 2}, {2, 2, 0, 1, 0, 0, 0, 0, 1, 1, 2},
    {2, 2, 0, 1, 1, 1, 1, 2}, {1, 1, 1, 0, 0, 0, 0, 1}, {1, 2, 1, 1, 0, 0, 0, 0, 1},
    {2, 1, 0, 0, 0, 0, 0, 1, 2, 2}, {3, 2, 2, 1, 1, 1, 1, 2, 3}},
   {{1, 1, 1, 1, 1, 1}, {2, 1, 2, 1, 2}, {2, 1, 1, 1, 1, 2}, {3, 2, 2, 1, 2, 2, 2, 2, 3},
    {2, 1, 1, 1, 1, 1, 1, 2}, {2, 1, 2, 1, 1, 1, 1, 2}, {1, 1, 1, 0, 0, 0, 0, 1},
    {1, 2, 1, 1, 0, 0, 0, 0, 1}, {2, 1, 1, 0, 0, 0, 0, 1, 1, 2},
    {2, 1, 1, 0, 0, 0, 0, 1, 1, 2}, {3, 2, 2, 1, 1, 1, 1, 2, 2, 3}},
   {{1, 1, 1, 1, 1}, {1, 0, 0, 1}, {2, 1, 0, 1, 2}, {2, 1, 0, 0, 1, 2},
    {3, 2, 1, 1, 2, 2, 2, 2, 3}, {2, 1, 0, 1, 1, 1, 1, 2},
    {1, 0, 0, 0, 0, 0, 1}, {1, 1, 1, 0, 0, 0, 0, 1}, {2, 1, 0, 0, 0, 0, 1, 2},
    {2, 1, 0, 0, 0, 0, 1, 2}, {3, 2, 1, 1, 1, 1, 2, 3}},
   {{2, 2, 2, 2}, {2, 0, 2}, {2, 0, 0, 2}, {3, 1, 1, 1, 3}, {3, 1, 1, 1, 1, 3},
    {4, 2, 2, 2, 3, 3, 3, 3, 4}, {2, 0, 0, 1, 1, 1, 1, 2}, {2, 2, 0, 1, 1, 1, 1, 2},
    {3, 1, 1, 1, 1, 3}, {3, 1, 1, 1, 1, 3}, {4, 2, 2, 2, 2, 4}},
   {{2, 0, 1, 2, 2, 2, 2}, {3, 1, 2, 2, 3, 3, 3, 3}, {3, 1, 1, 2, 2, 3, 3, 4, 3, 3},
    {3, 3, 4, 2, 2, 2, 3, 3}, {2, 2, 1, 1, 2, 2}, {4, 1, 2, 2, 2, 2, 4},
    {2, 0, 2, 2}, {2, 1, 1, 2}, {3, 2, 0, 1, 1, 1, 3, 3}, {3, 2, 2, 1, 1, 1, 1, 2, 3},
    {3, 1, 1, 2, 3, 3, 1, 1, 2}, {3, 1, 1, 3, 3, 1, 2, 1, 2}, {4, 2, 2, 2, 4, 3, 2, 3, 4}},
   {{1, 0, 0, 1, 1, 1, 1}, {2, 1, 0, 1, 2, 2, 3, 2, 2}, {2, 1, 0, 1, 1, 2, 2, 3, 2, 2},
    {2, 2, 2, 1, 1, 2, 2}, {1, 0, 0, 1, 1}, {2, 0, 0, 0, 0, 0, 1, 2}, {1, 0, 0, 1},
    {3, 2, 0, 1, 1, 1, 3, 3}, {2, 1, 0, 0, 0, 0, 0, 1, 2, 2}, {2, 1, 0, 0, 1, 2, 2, 0, 0, 1},
    {2, 1, 0, 0, 1, 2, 2, 0, 0, 1}, {3, 2, 1, 1, 3, 3, 1, 1, 1, 3}},
   {{1, 0, 0, 1, 1, 1, 2, 1, 1}, {2, 1, 1, 1, 2, 2, 2, 3, 2, 2},
    {2, 1, 1, 0, 1, 2, 2, 2, 3, 3, 2, 2}, {2, 2, 2, 2, 1, 1, 2, 2},
    {2, 1, 2, 1, 2, 2}, {1, 0, 0, 1, 1}, {2, 2, 0, 1, 0, 0, 0, 0, 1, 1, 2},
    {1, 0, 1, 0, 1}, {3, 2, 2, 1, 1, 1, 1, 2, 3}, {2, 1, 0, 0, 0, 0, 0, 1, 2, 2},
    {2, 1, 1, 0, 0, 1, 2, 2, 0, 0, 1}, {2, 1, 1, 0, 0, 1, 2, 2, 0, 0, 1},
    {3, 2, 2, 1, 1, 2, 3, 3, 1, 1, 2}}, {{1, 0, 0, 1, 2, 1, 1, 1, 1, 2},
    {2, 1, 1, 1, 2, 3, 2, 2, 2, 2, 3}, {2, 1, 1, 1, 1, 2, 3, 2, 2, 2, 2, 3},
    {2, 1, 1, 1, 2, 2, 2, 2, 3, 2, 2}, {1, 0, 0, 1, 1, 1, 1, 1, 1},
    {1, 0, 1, 1, 1, 0, 1}, {3, 1, 1, 1, 1, 3}, {2, 1, 0, 0, 0, 0, 1, 2},
    {2, 1, 1, 0, 0, 0, 0, 1, 1, 2}, {2, 1, 1, 0, 0, 1, 2, 2, 0, 0, 1},
    {2, 1, 0, 0, 1, 2, 2, 0, 0, 1}, {3, 1, 1, 2, 3, 3, 1, 1, 2}},
   {{1, 0, 0, 1, 1, 1, 1, 2, 1, 2}, {2, 1, 1, 1, 2, 3, 2, 2, 2, 2, 3},
    {2, 1, 1, 1, 1, 2, 3, 2, 2, 2, 2, 3}, {2, 1, 1, 1, 2, 2, 2, 2, 3, 2, 2},
    {1, 0, 0, 1, 1, 1, 1, 1, 1}, {1, 0, 1, 1, 1, 0, 1}, {3, 1, 1, 1, 1, 3},
    {2, 1, 0, 0, 0, 0, 1, 2}, {2, 1, 1, 0, 0, 0, 0, 1, 1, 2},
    {2, 1, 1, 0, 0, 1, 2, 2, 0, 0, 1}, {2, 1, 0, 0, 1, 2, 2, 0, 0, 1},
    {3, 1, 1, 3, 3, 1, 2, 1, 2}}, {{2, 1, 1, 2, 2, 2, 1, 2, 1, 2},
    {3, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3}, {3, 2, 2, 2, 2, 3, 4, 3, 3, 3, 3, 4},
    {3, 2, 2, 2, 3, 3, 3, 3, 4, 3, 3}, {2, 1, 1, 2, 2, 2, 2, 2, 2},
    {2, 1, 2, 2, 2, 1, 2}, {4, 2, 2, 2, 2, 4}, {3, 2, 1, 1, 1, 1, 2, 3},
    {3, 2, 2, 1, 1, 1, 1, 2, 2, 3}, {3, 2, 2, 1, 1, 2, 3, 3, 1, 1, 2},
    {3, 2, 1, 1, 3, 3, 1, 1, 1, 3}, {4, 2, 2, 2, 4, 3, 2, 3, 4}}}
```

```
In[143]:= tmp = % // Flatten; {Length[tmp], Union[tmp]}

Out[143]= {1278, {0, 1, 2, 3, 4}}
```