Reductivity properties over an affine base

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Some dates

- 1974: Arcata Conference on Algebraic Geometry.
  Bill Haboush, Mark Krusemeyer, Daniel Quillen.
- 1976: Northwestern University. Robert Griess tells about CPS.
- June 1976: I sit in meadow at Cornell University.
- Years later: Brian instructs me to try and prove finite generation.
- $\approx$1997: Friedlander and Suslin prove it.
- 2004: I admit defeat and Karen tells that Brian is jealous.
- January 2008: Vincent Franjou says Antoine Touzé is busy.
- 2009: my retirement.

Evidence of stability

Evanston 1976

Robert Griess: Let G be a reductive group in characteristc $p > 0$. Let V be one of its representations.
Examples show that $H^1(G, V)$, $H^2(G, V)$ stabilize as the size q of the field becomes large.

Do not confuse $H^i_{rat}(G, V)$ with $H^i(G(q), V)$.

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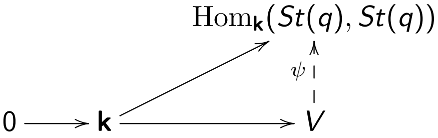
CPS: Mumford conjecture from Kempf vanishing

CPS: The height $h(\alpha)$ of a simple root α equals 1.
Let λ, μ be dominant.
If $H^n(B, -\mu \otimes \mathrm{ind}_B^G(-\lambda)) \neq 0$ then $h(\lambda - \mu) \geq n$.
So Kempf vanishing gives $H^1(B, -\mu \otimes \mathrm{ind}_B^G(-\lambda)) =$
$H^1(G, \mathrm{ind}_B^G(-\mu) \otimes \mathrm{ind}_B^G(-\lambda)) = H^1(B, -\lambda \otimes \mathrm{ind}_B^G(-\mu)) = 0$.

For $q \gg 0$ you get the lift ψ in



from

$$\mathrm{Ext}_G^1(V/\mathbf{k}, St(q)^* \otimes St(q)) \cong H^1(B, (V/\mathbf{k})^* \otimes -(q-1)\rho \otimes St(q)) = 0.$$

A twist

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- Years later: Brian instructs me to try and prove finite generation.

Now notice that

$$H^*(G_r, \mathbf{k}) \cong H^*(G, \mathbf{k}[G]^{(r)}).$$

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Finite generation of invariants

From now on G is an affine flat group scheme over a ring $\mathbf{k}$.

Theorem (Finite generation of invariants)

Let G be reductive in a suitable sense and let $\mathbf{k}$ be noetherian. If G acts on a finitely generated commutative $\mathbf{k}$-algebra A then A^G is a finitely generated commutative $\mathbf{k}$-algebra.

Over a field geometric reductivity is a basic notion, but our $\mathbf{k}$ is just a commutative noetherian ring.

A choice is needed

For Haboush geometric reductivity refers to any property that over a field is equivalent to his result.
If one intends to use a notion in a wider context, one must carefully select a phrasing.
That is why Vincent Franjou and I make a definition.
Go back to Mumford.

Power reductivity; a basic notion

Definition

The group G is *power reductive* over $\mathbf{k}$ if the following holds.

Property (Power reductivity)

Let $\varphi : M \twoheadrightarrow \mathbf{k}$ be a surjective map of G-modules. Then there is a positive integer d such that the d-th symmetric power of φ is a split surjection of G-modules

$$S^d\phi : S^dM \overset{\curvearrowleft}{\twoheadrightarrow} S^d\mathbf{k}.$$

In other words, one requires that the kernel of $S^d\phi$ thas a complement in S^dM.

The conjecture of Mumford

Mumford conjectured in the introduction to the first edition of his GIT book that a semisimple algebraic group defined over a field of positive characteristic p is power reductive.

He also stipulated that the integer d may be taken a power of p.
It turns out that this already follows from power reductivity when the base ring has characteristic p.
We have no use for p in our general context.

Notice that $S^*\mathbf{k} \cong \mathbf{k}[T]$, and we have a graded map $S^*M \twoheadrightarrow \mathbf{k}[T]$.
Looking for a splitting is trying to lift a *power* of T to an invariant.

Geometric reductivity after Seshadri

Let the representation V be finitely generated and free as a $\mathbf{k}$-module.
Then it makes sense to speak of polynomial functions on V.
Suppose one has a nonzero invariant vector v at some geometric point:

$$0 \neq v \in (V \otimes_{\mathbf{k}} \kappa)^G.$$

Then Seshadri asks for an invariant homogeneous polynomial function F of positive degree on V which does not vanish at v.
Geometric reductivity is now the property that such F always exists.

No longer a basic notion

It has been known for a long time that in representation theory the group should be flat, but that modules should not be restricted that way.
The fact that geometric reductivity restricts attention to modules that are free over $\mathbf{k}$ makes that geometric reductivity is not the right notion over an affine base.
When Nagata proves finite generation of a ring of invariants he factors out an arbitrary homogeneous G-invariant ideal in a graded algebra. This may produce a non-flat algebra.

Equivariant flat resolutions

The unfortunate restriction to free modules explains why Seshadri runs into the problem of the existence of equivariant flat resolutions when he tries to prove finite generation of invariants. The existence of equivariant flat resolutions may be an interesting problem in itself, but it has little to do with invariant theory.

Exception. Recall that for a flat affine G over a *discrete valuation ring* (DVR) one does have equivariant free resolutions and local finiteness of representations [Serre 1968]. Nothing more about G is needed in this particular case.

Power reductivity as a basic notion

- Power reductivity implies geometric reductivity.
- If **k** is noetherian then power reductivity is *equivalent* to the theorem of finite generation of invariants.

Let us call a map of commutative rings $\phi : A \to B$ *power surjective* if for every $b \in B$ there is some positive power b^n in the image of ϕ.

The following are equivalent:

- G is power reductive,
- For every power surjective equivariant $\phi : A \to B$ the map $A^G \to B^G$ is power surjective,
- B^G is integral over $\phi(A^G)$ when $\phi : A \twoheadrightarrow B$ is surjective.

Examples

One concludes:

- Finite group schemes are power reductive.

(A is integral over A^G, so $B^G \subseteq \phi(A)$ is integral over $\phi(A^G)$.)

- Extensions of power reductive group schemes by power reductive group schemes are power reductive.

(If $N \triangleleft G$ and G/N exists as an affine flat group scheme, then $V^G = (V^N)^{G/N}$.)

- If G is power reductive, then $A^G \to (A/pA)^G$ is power surjective [in fact 'p-power surjective'] for every prime p.

Base change

If $\mathbf{k} \to L$ is a map of commutative rings and V is a G_L-module, then it is also a G-module and $H^*(G_L, V) = H^*(G, V)$.
In particular [reference?]

$$V^{G_L} = V^G.$$

Therefore:

- Power reductivity is preserved by base change.

As integrality descends along faithfully flat maps,

- power reductivity descends along faithfully flat maps.

For similar reasons:

- Power reductivity is Zariski local.
- It suffices to have power reductivity for the local rings $\mathbf{k}_{\mathfrak{m}}$.

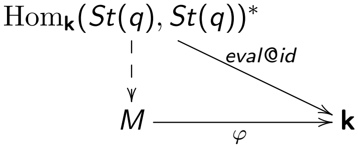
Reductive group scheme

So for a reductive group scheme in the sense of SGA3 (smooth over **k** with geometric fibers that are connected reductive) it suffices to consider the case where G is split and defined over a discrete valuation ring **k**.
But over a discrete valuation ring power reductivity is equivalent to geometric reductivity. (In his 1968 paper on representation rings Serre has taken care of local finiteness and equivariant free resolutions for such **k**.)
Thus by Seshadri (1977)

- Reductive group schemes are power reductive.

CPS style proof of power reductivity

Say **k** is a discrete valuation ring and G is split reductive.
Let $\phi : M \twoheadrightarrow \mathbf{k}$.
To find a splitting of some $S^d\phi$ we may assume M is finitely generated free **k**-module [Serre 1986].
Now use $\mathrm{Ext}^1_G(St(q) \otimes St(q), \ker(M \twoheadrightarrow \mathbf{k})) = 0$ for $q \gg 0$ to get



Notice that $\mathrm{Ext}^1_G(St(q) \otimes St(q), \cdots)$ is much better behaved than $\mathrm{Ext}^1_G(\cdots, St(q) \otimes St(q))$.

Query

Recall G is affine and flat over $\mathbf{k}$.
If $\mathbf{k}$ is noetherian then representations are locally finite [Serre 1968] and also if $\mathbf{k}[G]$ is a projective $\mathbf{k}$-module [Seshadri 1977].
In the book of Jantzen (2003) it is claimed that 'we know that' an intersection of sub[-co]modules is a sub[-co]module.
For a finite intersection this is correct, but otherwise it already fails over a discrete valuation ring (Serre, Gabber [SGA3 2011]).
One cannot speak of the smallest sub[-co]module containing a given vector.
So the proof in the book of local finiteness is all wrong.
Are there counter-examples to local finiteness?

Take it easy !

The Serre–Gabber example

Let $\mathbf{k}$ be a DVR, K its field of fractions.
Put $\mathbf{k}[G] = \{ P \in K[T] \mid P(0) \in \mathbf{k} \}$, with $\Delta(T) = T \otimes 1 + 1 \otimes T$, so that $G_K = \mathbb{G}_{a,K}$. Let

$$u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^2, \mathbf{k}^2),$$

$$\Delta(m) = 1 \otimes m + T \otimes u(m).$$

The submodules N of $\mathbf{k}^2$ that contain $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ have an intersection that is not a submodule.
If N is a submodule containing $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then N is of the form $\mathbf{k} \oplus I$, where I is a nonzero ideal in $\mathbf{k}$. Indeed $\Delta N \subseteq \mathbf{k}[G] \oplus I\mathbf{k}[G]$ gives $\begin{pmatrix} 1 \\ T \end{pmatrix} \in \mathbf{k}[G] \oplus I\mathbf{k}[G]$ so that $I \neq 0$.