Bifunctors and Formality

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Upcoming slides

- Schur algebras
- Schur category
- Functor category
- Bifunctors
- Cohomology
- Yoneda Lemma
- Derived categories
- Touzé classes
- Chałupnik formality for Ext
- Chałupnik formality for bifunctor cohomology
- Formality with Hopf algebra objects

Schur algebras

k is a field of characteristic $p > 0$.
$\Gamma^d M := (M^{\otimes d})^{\mathfrak{S}_d} = (S^d(M^\vee))^\vee$.
The Schur algebra $S(n, d)$ is $\Gamma^d \operatorname{Hom}_k(k^n, k^n)$.
rep $S(n, d)$ is equivalent to the category of
polynomial representations of degree d of $\mathrm{GL}(n)$.
All categories and many functors will be k-linear or, in categorical jargon, enriched in k-*Mod*.
An algebra is a k-linear category A with one object.
An A-module M is a k-linear functor from A to k-*Mod*.
Rep A is the category of A-modules.
rep A is the category of k-linear functors from A to k-*mod*.

Schur category

The Schur category $S(d)$ of degree d is the k-linear category whose objects are finite dimensional vector spaces and whose morphisms are given by

$$\mathrm{Hom}_{S(d)}(V,W) := \Gamma^d \mathrm{Hom}_k(V,W) = \Gamma^d(V^\vee \otimes_k W).$$

So the Schur category of degree d contains the Schur algebras $S(n,d)$ as full subcategories.
Composition satisfies $f^{\otimes d} \circ g^{\otimes d} = (f \circ g)^{\otimes d}$.
One shows that the composition maps

$$\mathrm{Hom}_{S(d)}(V,k^n) \otimes_k \mathrm{Hom}_{S(d)}(k^n,W) \to \mathrm{Hom}_{S(d)}(V,W)$$

are surjective for $n \geq d$.
From this it follows arXiv:1103.4580 that the restriction $rep\, S(d) \to rep\, S(n,d)$ is an equivalence of categories for $n \geq d$.

Functor category

Functor categories like $rep\, S(d)$ come with a rich toolkit.

The Yoneda Lemma provides enough injectives and projectives in $rep\, S(d)$.

Adjunction between 'sum' and 'diagonal' allows to break up many Ext groups. (keywords: exponential functors.)

To an element F of $rep\, S(d)$ one associates what Friedlander and Suslin call a strict polynomial functor of degree d. It sends $f \in \mathrm{Hom}_k(V, W)$ to $F(f^{\otimes d})$. Their description is different. They use functors between categories of finite dimensional vector spaces enriched over a category of affine varieties.

Our notation often refers to the strict polynomial functors, but we argue with $rep\, S(d)$.

Composition of $F \in rep\, S(d)$ with $G \in rep\, S(e)$ satisfies $(F \circ G)f^{\otimes de} = F(G(f^{\otimes e})^{\otimes d})$.

Bifunctors

We will be interested in functors like $\Gamma^d \operatorname{Hom}(-_1, -_2)$.
Let us write the category of such bifunctors as $rep\, S(d)^{\mathrm{opp}} \otimes S(d)$.
Here $\operatorname{Hom}_{S(d)^{\mathrm{opp}} \otimes S(d)}((V, X), (W, Y)) :=$
$\Gamma^d(\operatorname{Hom}_k(W, V)) \otimes \Gamma^d(\operatorname{Hom}_k(X, Y))$.
A bifunctor restricts to a $S(n, d)$-bimodule.
A key example is $\Gamma^d \mathfrak{gl}^{(1)} : (V, W) \mapsto \Gamma^d(\operatorname{Hom}_k(V^{(1)}, W^{(1)}))$.
It is of bidegree (pd, pd).
Here (1) denotes precomposition by the Frobenius twist functor $I^{(1)} := \ker(S^p \to \Gamma^p) = \operatorname{coker}(S^p \to \Gamma^p)$.
If one takes $V = W$ in $\Gamma^d(\operatorname{Hom}_k(V^{(1)}, W^{(1)}))$, one gets the representation $\Gamma^d \mathfrak{gl}^{(1)}$ of $\mathrm{GL}(V)$. It is needed in the proof of my cohomological finite generation conjecture. This was my initial reason to care about the representation/bifunctor $\Gamma^d \mathfrak{gl}^{(1)}$.

Cohomology

Let $\dim V \geq d$.
The functor from *rep* $S(d)^{\mathrm{opp}} \otimes S(d)$ to *k-mod* which sends F to $F(V,V)^{\mathrm{GL}(V)}$ is left exact and thus representable.
The representing object is the bifunctor $\Gamma^d \mathfrak{gl}$.
Using good filtration theory this implies an isomorphism

$$\mathrm{Ext}^i(\Gamma^d \mathfrak{gl}, F) \cong H^i\left(\mathrm{GL}(V), F(V,V)\right).$$

That is why we put $H^i(F) := \mathrm{Ext}^i(\Gamma^d \mathfrak{gl}, F)$ and call it the i-th functor cohomology of the bifunctor F.
Our theme is the interaction between cohomology and Frobenius twist.
The main result will be a formality theorem implying that Frobenius twist causes an extra grading on Ext groups.

The Yoneda Lemma in *rep* $S(d)$ takes the form

$$\mathrm{Hom}_X\left(\Gamma^d(Y^\vee \otimes X), F(X)\right) \simeq F(Y),$$

with notations analogous to

$$\int_{x=-\pi}^{\pi} f(x,y)\,dx = \phi(y).$$

So the bound variable X indicates with respect to which Schur algebra/category the Hom is to be taken. The Yoneda Lemma generalizes to $\mathrm{Hom}_X(\Gamma^d(H^\vee \otimes X), F(X)) \simeq F \circ H$ where H is a strict polynomial functor.
For F in *rep* $S(d)^{\mathrm{opp}}$ it becomes

$$\mathrm{Hom}^X\left(\Gamma^d(H \otimes X^\vee), F(X)\right) \simeq F \circ H.$$

Derived categories

Say $\mathcal{A}$ is an abelian category. For an object F of $\mathcal{A}$ we denote by $F[-n]$ the corresponding cochain complex $\cdots \to 0 \to F \to 0 \cdots$ concentrated in cohomological degree n.
We also view it as a graded object with F in degree n, or as an object with $\mathbb{G}_m$ action of weight n.
One has the following fundamental connection between Yoneda classes and some morphisms in the derived category $\mathcal{D}(\mathcal{A})$

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(F[-m], G[-n]) \cong \mathrm{Ext}^{m-n}_{\mathcal{A}}(F, G).$$

So some morphisms in the derived category can be understood as extension classes and *vice versa*.
We call an object C of $\mathcal{D}(\mathcal{A})$ formal if one is given an isomorphism

$$C \cong \bigoplus_i H^i(C)[-i].$$

Touzé classes

Antoine Touzé has constructed certain classes
$c[d] \in H^{2d}(\Gamma^d(\mathfrak{gl}^{(1)}))$.
The class $c[1]$ corresponds with $0 \to I^{(1)} \to S^p \to \Gamma^p \to I^{(1)} \to 0$
in $\mathrm{Ext}^2_X(X^{(1)}, X^{(1)}) \cong H^2(\mathfrak{gl}^{(1)})$.
For $d > 1$ the class $c[d]$ lifts $c[1]^{\cup d} \in H^{2d}(\bigotimes^d(\mathfrak{gl}^{(1)}))$.
Playing with the Yoneda Lemma Marcin Chałupnik showed that

$$H^i\left(\Gamma^d(\mathfrak{gl}^{(1)})\right) \cong \mathrm{Ext}^i\left(\Gamma^d(I^\vee \otimes I^{(1)}), \Gamma^d(I^\vee \otimes I^{(1)})\right).$$

So now the $c[d]$ define morphisms
$\Gamma^d(I^\vee \otimes I^{(1)})[-n-2d] \to \Gamma^d(I^\vee \otimes I^{(1)})[-n]$, or dually
$c[d] : S^d(I^\vee \otimes I^{(1)})[n] \to S^d(I^\vee \otimes I^{(1)})[n+2d]$
in $\mathcal{D}(rep\, S(d)^{\mathrm{opp}} \otimes S(dp))$.
They can be composed [Yoneda], leading to morphisms
$c[d]^i : S^d((I^\vee \otimes I^{(1)})[-2i]) \to S^d(I^\vee \otimes I^{(1)})$.

Chałupnik formality for Ext

Combining the $c[d]^i$ for various values of d and i, one gets

$$\alpha_S : S^d(Z^\vee \otimes E_1 \otimes X^{(1)}) \to S^d(Z^\vee \otimes X^{(1)})$$

where $E_1 := A_1 := \bigoplus_{i=0}^{p-1} k[-2i]$ may be thought of as a graded vector space, or a $\mathbb{G}_m$-module, or an element of a derived category. From this one constructs a morphism

$$\beta_S : S^d(Y \otimes E_1 \otimes Z^\vee) \to \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes X^{(1)}))$$

which turns out to be an isomorphism in $\mathcal{D}(rep\, S(d)^\vee \otimes S(d))$.
So $\mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes X^{(1)}))$ is formal.
One now uses β_S to show for F, G in $rep\, S(d)$ that

$$\mathrm{Ext}^n(F^{(1)}, G^{(1)}) \cong \bigoplus_{i+j=n} \mathrm{Ext}^i_X\,(F(X), G(X \otimes E_1)^j).$$

Here $G(X \otimes E_1)^j$ is the weight j component for the $\mathbb{G}_m$ action.

Chałupnik formality for bifunctor cohomology

So Ext groups between Frobenius twisted representations break up in terms of Ext groups between less twisted representations.
For a bifunctor B in $rep\, S(d)^{\mathrm{opp}} \otimes S(d)$ the result is similar

$$H^n\left(B(-_1^{(1)}, -_2^{(1)})\right) \cong \bigoplus_{i+j=n} H^i\left(B(-_1, -_2 \otimes E_1)^j\right).$$

Here $B(-_1, -_2 \otimes E_1)^j$ is the weight j component of $B(-_1, -_2 \otimes E_1)$ for the $\mathbb{G}_m$ action.
Chałupnik gives his results at the level of derived categories. So he uses RHom instead of Ext, as one should. Thus

$$\mathrm{RHom}_X(F(X^{(1)}), G(X^{(1)})) \cong \mathrm{RHom}_X\left(F(X), G(X \otimes E_1)\right),$$

and

$$\mathrm{RHom}_Y^X(\Gamma^{pd}\mathfrak{gl}(X,Y), B(X^{(1)}, Y^{(1)})) \cong \mathrm{RHom}_Y^X\left(\Gamma^d\mathfrak{gl}(X,Y), B(X, Y \otimes E_1)\right).$$

He also studies precomposition by $I^{(1)}$ as a functor with functors as arguments.

Formality with Hopf algebra objects

The existence of Touzé classes is equivalent to the formality theorem for bifunctor cohomology. Indeed the weight space $H^{2d}(\Gamma^d \mathfrak{gl}^{(1)})^{2d}$ maps isomorphically to $H^{2d}(\bigotimes^d \mathfrak{gl}^{(1)})^{\mathfrak{S}_d}$.
As Touzé has more than one construction of 'universal classes' $c[d]$, one must choose which classes to take.
We recommend the original construction.
In a category $\mathcal{D}(\prod_{d\geq 0} rep\, S(d)^\vee \otimes S(d))$ one gets with this choice of $c[d]$ an isomorphism of Hopf algebra objects

$$\bigoplus_{d\geq 0} S^d(Y \otimes E_1 \otimes Z^\vee)$$

$$\cong$$

$$\mathrm{RHom}_X\left(\bigoplus_{d\geq 0} \Gamma^d(Y^\vee \otimes X^{(1)}), \bigoplus_{d\geq 0} S^d(Z^\vee \otimes X^{(1)})\right).$$

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The End

Thank you !

Scratch slides

From functor formality to bifunctor formality:
$H(G(-_1^{(1)}, -_2^{(1)}))$ is (co)homology of
$\mathrm{RHom}_Y^X(\Gamma^{dp}(X^\vee \otimes Y), G(X^{(1)}, Y^{(1)})) \simeq$
$\mathrm{RHom}_Y^X(\Gamma^{dp}(X^\vee \otimes Y), \mathrm{RHom}^Z(\Gamma^d(Z^\vee \otimes X^{(1)}), G(Z, Y^{(1)}))) \simeq$
$\mathrm{RHom}_Y^Z(\Gamma^d(Z^\vee \otimes Y^{(1)}), G(Z, Y^{(1)})) \simeq$
$\mathrm{RHom}_X^Z(\Gamma^d(Z^\vee \otimes X), \mathrm{RHom}_Y(\Gamma^d(X^\vee \otimes Y^{(1)}), G(Z, Y^{(1)}))) \simeq$
$\mathrm{RHom}_X^Z(\Gamma^d(Z^\vee \otimes X), \mathrm{RHom}_Y(\Gamma^d(X^\vee \otimes Y), G(Z, Y \otimes E_1))) \simeq$
$\mathrm{RHom}_X^Z(\Gamma^d(Z^\vee \otimes X), G(Z, X \otimes E_1))$ with (co)homology
$H(G(-_1, -_2 \otimes E_1))$.

We have a map $\alpha_S : S^d(Z^\vee \otimes E_1 \otimes X^{(1)\vee}) \to S^d(Z^\vee \otimes X^{(1)^\vee})$.
We wish to show that the composite map
$\beta_S : S^d(Y \otimes E_1 \otimes Z^\vee) \simeq \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X), S^d(Z^\vee \otimes E_1 \otimes X)) \to \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes E_1 \otimes X^{(1)})) \to \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes X^{(1)}))$ is a quasi-isomorphism.
It probably induces the ϕ of Chałupnik.
First apply $\mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X), -)$ to the commutative diagram [lifted classes property]

$$\begin{array}{ccc} \bigotimes^d(Z^\vee \otimes E_1 \otimes X^{(1)}) & \xrightarrow{\alpha_\otimes} & \bigotimes^d(Z^\vee \otimes X^{(1)}) \\ \downarrow & & \downarrow \\ S^d(Z^\vee \otimes E_1 \otimes X^{(1)}) & \xrightarrow{\alpha_S} & S^d(Z^\vee \otimes X^{(1)}) \end{array}$$

in a derived category of multifunctors. (One always uses the exponential property in such setting.)

Then use the resulting square to construct a commuting square

$$\begin{array}{ccc} \bigotimes^d(Y \otimes E_1 \otimes Z^\vee) & \xrightarrow{\beta_\otimes} & \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), \bigotimes^d(Z^\vee \otimes X^{(1)})) \\ \downarrow & & \downarrow \\ S^d(Y \otimes E_1 \otimes Z^\vee) & \xrightarrow{\beta_S} & \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes X^{(1)})). \end{array}$$

By the exponential property, the top arrow is a quasi-isomorphism. The right arrow is surjective on cohomology by the collapsing spectral sequence, for instance. See also Theorem 4.6 in [Touzé, Troesch complexes ...], or section 6 of [Chałupnik, Extensions ...]. For given Y, Z the bottom arrow goes between objects with the same dimension of cohomology. Done.

Once one has the quasi-iso β_S, one may compute like this, ignoring sign issues.
$\mathrm{RHom}_X(F(X^{(1)}), G(X^{(1)})) \cong$
$\mathrm{RHom}^X(G^\vee(X^{(1)}), F^\vee(X^{(1)})) \cong$
$\mathrm{RHom}_X(G^\vee(X^{(1)}), \mathrm{RHom}^Y(\Gamma^d(Y^\vee \otimes X^{(1)}), F^\vee(Y))) \cong$
$\mathrm{RHom}_Y(F(Y), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), G(X^{(1)}))) \cong$
$\mathrm{RHom}_Y(F(Y), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes$
$X^{(1)}), \mathrm{RHom}^Z(G^\vee(Z), S^d(Z^\vee \otimes X^{(1)})))) \cong$
$\mathrm{RHom}_Y(F(Y), \mathrm{RHom}^Z(G^\vee(Z), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes$
$X^{(1)}), S^d(Z^\vee \otimes X^{(1)})))) \cong$
$\mathrm{RHom}_Y(F(Y), \mathrm{RHom}^Z(G^\vee(Z), S^d(Y \otimes E_1 \otimes Z^\vee))) \cong$
$\mathrm{RHom}_Y(F(Y), G(Y \otimes E_1)))$
And all this is functorial in F, G, so we may put in more variables to get the result for bifunctors.

As for sign issues, if C is a cochain complex define the differential $d^\vee$ on $C^\vee$ to be given by $d^\vee\phi = (-1)^{|\phi|+1}\phi \circ d$.
(Think $C^\vee = \mathrm{RHom}_k(C, k)$.)
Then $d^{\vee\vee} = -d$ and $\mathrm{Hom}^\bullet_k(C, D) = \mathrm{Tot}(D \otimes_k C^\vee)$.
Also $\mathrm{Tot}(C \otimes_k D)^\vee = \mathrm{Hom}^\bullet_k(C, D^\vee) = \mathrm{Tot}(D^\vee \otimes_k C^\vee)$.
But $\mathrm{Tot}(C \otimes_k D) \sim \mathrm{Tot}(D \otimes_k C)$ involves braiding signs $(-1)^{rs}$ at $C^r \otimes D^s$.
When thinking about signs in multicomplexes the case of vector spaces may serve as a guide.
The Yoneda lemma becomes
$\mathrm{RHom}_X\left(\Gamma^d(Y^\vee \otimes X), F^\bullet(X)\right) \simeq F^\bullet(Y)$, or dually
$\mathrm{RHom}^X\left(F^\bullet(X), S^d(Y \otimes X^\vee)\right) \simeq (F^\bullet(Y))^\vee$.

Taking all this into account we compute $\mathrm{RHom}_X(F(X^{(1)}), G(X^{(1)})) \sim \mathrm{RHom}^X(G^{\vee\vee\vee}(X^{(1)}), F^\vee(X^{(1)})) \cong \mathrm{RHom}^X(G^{\vee\vee\vee}(X^{(1)}), \mathrm{RHom}^Y(\mathrm{RHom}_Y(\Gamma^d(Y^\vee \otimes X^{(1)}), F^\vee(Y)))) \sim \mathrm{RHom}_Y(F(Y), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), G(X^{(1)}))) \cong \mathrm{RHom}_Y(F(Y), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), \mathrm{RHom}^Z(G^{\vee\vee\vee}(Z), S^d(Z^\vee \otimes X^{(1)})))) \sim \mathrm{RHom}_Y(F(Y), \mathrm{RHom}^Z(G^{\vee\vee\vee}(Z), \mathrm{RHom}_X(\Gamma^d(Y^\vee \otimes X^{(1)}), S^d(Z^\vee \otimes X^{(1)})))) \cong \mathrm{RHom}_Y(F(Y), \mathrm{RHom}^Z(G^{\vee\vee\vee}(Z), S^d(Y \otimes E_1 \otimes Z^\vee))) \cong \mathrm{RHom}_Y(F(Y), G(Y \otimes E_1)))$, with some "extra braiding signs" at the $\sim$ steps.

Now with notations as in my lectures on Bifunctors ...
Let $R : rep\, S(d) \to k\text{-}Mod$ be a left exact functor. Then put $L(X) := R(S^d_X)$. One has $\mathrm{Hom}_X(L^\#(X), S^d_V) \cong R(S^d_V)$ functorially in S^d_V. So R is representable.
Similarly, $B \mapsto B(V,V)^{\mathrm{GL}(V)}$ is representable by L with $L^\#(X,Y) = \mathrm{Hom}(\Gamma^d_X V, S^d_Y V)^{\mathrm{GL}(V)} = \mathrm{Hom}_V(\Gamma^d_X V, S^d_Y V) = S^d_Y(X^\vee) = S^d\mathfrak{gl}(X,Y)$.