Columbus,

Ohio, 2015

Universiteit Utrecht

Extrapolating an Euler class

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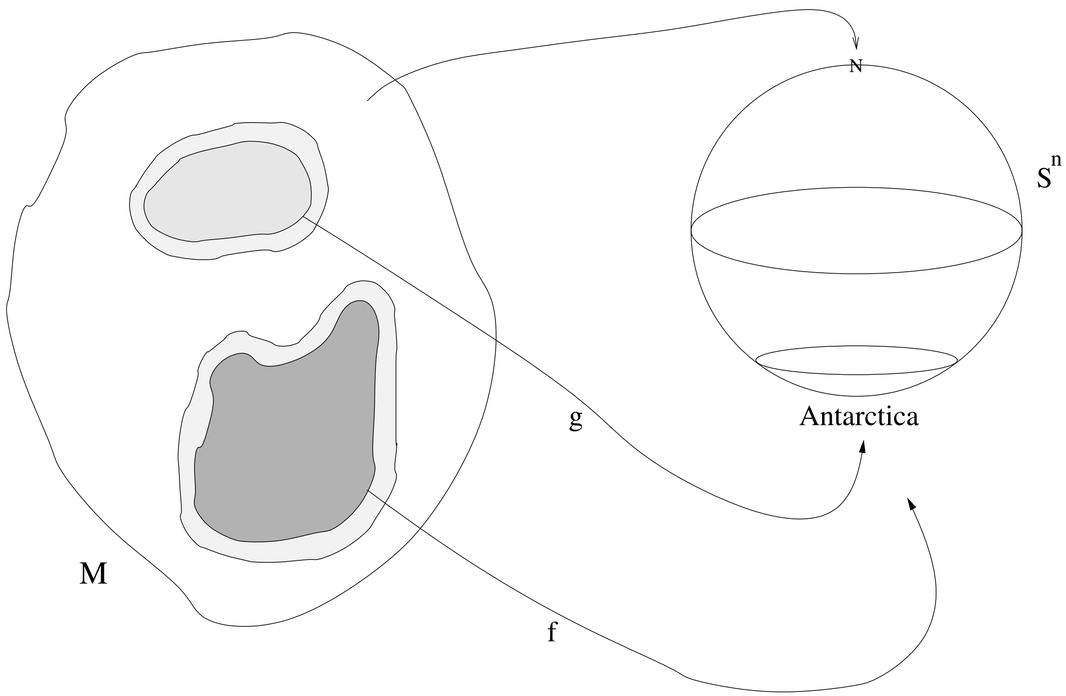
Borsuk addition on the cohomotopy group $[M, S^n]$, for $\dim(M) = d \leq 2n - 2$.



One must arrange that f^{-1}(Antarctica) and g^{-1}(Antarctica) are disjoint.

The Borsuk representative of $[f] + [g]$ then agrees with f on f^{-1}(Antarctica) and with g on g^{-1}(Antarctica).

It sends nothing else to Antarctica and sends most of M to the North pole.
[Borsuk 1961/62, 1937]

Let us shrink Antarctica towards the South pole. We may assume f is smooth around the fiber of the South pole and that the South pole is a regular value.

To know the homotopy class of f, it suffices to know the map from the normal bundle of f^{-1}(South pole) to the tangent space at the South pole. So the class of f is given by a *trivialization of the conormal bundle* of a *codimension* n *subvariety*.

The weak Mennicke symbol group WMS_{n+1}.

Let R be a commutative noetherian ring of Krull dimension d, $d \leq 2n-2$. Think of R as the ring continuous real valued functions on M. (The analogy is that both R and this ring of functions satisfy certain Bass' stable range conditions. We say they have stable range dimension at most d.)

Think of an element of $\text{Um}_{n+1}(R)$ as a map from M to $\mathbb{R}^{n+1} \backslash \{0\}$ and contract $\mathbb{R}^{n+1} \backslash \{0\}$ to the n-sphere.

Think of an element of the orbit set $\text{Um}_{n+1}(R)/E_{n+1}(R)$ as a homotopy class of maps $M \to S^n$.

One shows that given two homotopy classes (orbits) there are respective representatives

$(a_1, \ldots, a_{n+1})$, $(b_1, \ldots, b_{n+1})$ so that

$$a_1 + b_1 = 1 \text{ and } a_i = b_i \text{ for } i > 1.$$

One defines the sum of the two orbits to be the orbit of $(a_1 b_1, a_2, \ldots, a_{n+1})$. (*Weak Mennicke symbol rule.*)

One is still waiting for a *nice* proof that this is well-defined. It is well-defined [vdK 1989], and it defines an abelian group structure on $WMS_{n+1}(R) = \mathsf{Um}_{n+1}(R)/E_{n+1}(R)$.

Conceptually the addition agrees with Borsuk, thanks to the

Lemma (Ofer Gabber) [Paris, at lunch]
If $x + y = 1$, then x and y are not both negative. $\square$

Thus for representatives as above, the two inverse images of Antarctica are disjoint, and the Borsuk sum is easily seen to be the homotopy class given by $(a_1 b_1, a_2, \ldots, a_{n+1})$.

EULER CLASS GROUPS.

From now on assume $3 \leq d \leq 2n-3$,

cf. [Bhatwadekar-Sridharan 2000].

The n-th Euler class group $E^n(R)$ is defined by a presentation.

Generators

Pairs (J, ω_J) where J is a height n ideal in R equipped with a surjective map $(R/J)^n \to J/J^2$. Think of a codimension n subvariety with trivial conormal bundle together with a trivialization of said bundle.

Relations

- **Disconnected sum relation**

Let (J, ω_J) be a generator. If $J = KL$ with K, L comaximal ideals of height n, then $R/J = R/K \times R/L$ and $J/J^2 = K/K^2 \times L/L^2$, so that $\omega_J = \omega_k \times \omega_L$. The relation is

$$(J, \omega_J) = (K, \omega_K) + (L, \omega_L).$$

• **Complete intersection relation**

Let (J, ω_J) be a generator such that ω_J lifts to a surjection $R^n \to J$. Then

$$(J, \omega_J) = 0.$$

• **Elementary action relation** (implied by the above relations, already when $2 \leq d \leq 2n-1$.) Let (J, ω_J) be a generator and let $g \in E_n(R/J)$. Then

$$(J, \omega_J) = (J, \omega_J \circ g).$$

The problem

Proposal of Jean Fasel.

Given a unimodular row $(a_1, \ldots, a_{n+1})$ for which the ideal $J = (a_1, \ldots, a_n)$ has height n, attach to it the Euler class of (J, ω_J), where ω_J is given by $(\bar{a}_1, \ldots, \bar{a}_{n-1}, \bar{a}_n \bar{a}_{n+1})$.

Question Does this define a homomorphism $WMS_{n+1}(R) \to E^n(R)$?

Theorem Yes, if R contains an infinite field F.

Remark. There are several related results in the literature [Bhatwadekar-Sridharan 2000; Das-Zinna 2015]. Our proof relies on the

Known Fact [vdK 1977]
Every Zariski open subset of $SL_m(F)$ is path connected for walks in which a step is right multiplication by an elementary matrix.

Sketch of proof of the Theorem

Let us call the unimodular row $(a_1, \ldots, a_{n+1})$ *generic* if both a_n and a_{n+1} avoid every minimal prime ideal. Let $\mathrm{Um}_{\mathrm{gen}}$ be the set of such rows.

Consider $(a_1, \ldots, a_{n+1}) \in \mathrm{Um}_{\mathrm{gen}}$. After adding suitable multiples of a_{n+1} to a_1, ..., a_{n-1} the rule of Fasel gives an Euler class denoted $\phi(a_1, \ldots, a_{n+1})$.

If instead we add suitable multiples of a_n to a_1, ..., a_{n-1}, and subsequently interchange a_n, a_{n+1}, we get another Euler class. These two classes can be added by the disconnected sum relation and the result vanishes by the complete intersection relation. It follows that $\phi(a_1, \ldots, a_{n+1})$ is well-defined.

We now wish to extend ϕ from $\mathsf{Um}_{\mathsf{gen}}$ to all of $\mathsf{Um}_{n+1}(R)$.

One checks by computation that ϕ is constant along walks that stay inside $\mathrm{Um}_{\mathrm{gen}}$.

Using the Fact above, one sees that every $SL_{n+1}(F)$-orbit in $\mathrm{Um}_{n+1}(R)$ intersects $\mathrm{Um}_{\mathrm{gen}}$ in a path connected subset. So now ϕ can be extended to all of $\mathrm{Um}_{n+1}(R)$ by requiring that ϕ is constant on $SL_{n+1}(F)$-orbits. One checks that if P is a path component of $\mathrm{Um}_{\mathrm{gen}}$, then the union of the $SL_{n+1}(F)$-orbits of elements of P is an $E_{n+1}(R)$-orbit.

So we get a map $WMS_{n+1}(R) \to E^n(R)$. It is a homomorphism by the disconnected sum relation. $\square$

We needed the assumption $d \leq 2n-3$ to stay in the range where Bhatwadekar and Sridharan have defined $E^n(R)$.

If we simply define $E^n(R)$ for $d \leq 2n-1$ by the same presentation as above, then we get a map $\mathrm{Um}_{n+1}(R)/E_{n+1}(R) \to E^n(R)$ for $3 \le n \le d \le 2n-1$.

But for the group structure on the orbit set one needs $d \le 2n-2$, just like Borsuk.

The map $WMS_{n+1}(R) \to E^n(R)$ will be a homomorphism if $3 \le n \le d \le 2n-2$.

Our reasoning fails for $n = d = 2$.