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Homology Stability for Linear Groups

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Summary. Let R be a commutative finite dimensional noetherian ring or, more generally, an associative ring which satisfies one of Bass' stable range conditions. We describe a modified version of H. Maazen's work [18], yielding stability for the homology of linear groups over R. Applying W.G. Dwyer's arguments (cf. [9]) we also get stability for homology with twisted coefficients. For example, $H_2(GL_n(R), R^n)$ takes on a stable value when n becomes large.

§ 1. Introduction

1.1. Our motivation for this work has been to prove stability for algebraic K-theory in BGL^+ context. Thanks to the recent work of Dwyer we actually get much more general statements. These imply a result which seems to be of interest to geometric topologists. Namely, we find that the twisted homology groups $H_k(GL_n(R), \rho_n)$, considered by Dwyer in [9], stabilize with respect to n not only when R is a PID, but also when R is the group ring $\mathbb{Z}[\pi]$ of a finite group π. This fits in with work of W.G. Dwyer, Wu-Chung Hsiang and R.E. Staffeldt on Waldhausen's rational algebraic K-groups of a space.

1.2. Let us remind the reader what sort of families $\{\rho_n\}$ are considered by Dwyer, leaving out all technicalities and using some suggestive but unexplained terminology. A basic example is the family $\lambda = \{\lambda_n\}$, where λ_n denotes (the standard representation of $GL_n(R)$ in) the right R-module R^n of column vectors of length n over R. This system λ grows linearly with n. Note that the difference between λ_{n+1} and λ_n is equal to R for all n, so that the system of differences is constant in this case. More generally Dwyer considers systems that grow polynomially with n, such as the system $\mu = \{\mu_n\}$, where μ_n denotes (the representation by conjugation of $GL_n(R)$ in) the space of n by n matrices over R. The system μ grows quadratically with n, which can be rephrased by saying that its system of third iterated differences is zero, while its system of second iterated differences is not zero. (To make sense of all this, one has to add more structure

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to the data, and one has to be careful about the way the system of differences of a system ρ inherits its structure from ρ. See [9] and Sect. 5 below.) Now our main result is as follows. Let R be a commutative noetherian ring of finite Krull dimension or, more generally, an associative ring (with identity) which satisfies one of Bass' stable range conditions SR_m (cf. [2], Chap. V, Thm. 3.5). Let $\rho=\{\rho_n\}$ be a system as in Dwyer [9]. Then for fixed k the homology groups $H_k(GL_n(R), \rho_n)$ assume a constant value when n becomes large. The proof also works with $GL_n(R)$ replaced by its elementary subgroup $E_n(R)$ and this is how we get stability for the "unstable" K-groups $\pi_m(BGL_n^+(R))$, with reasonable estimates for the range of stability. (See Corollary 4.12 below.)

1.3. We now wish to discuss some preceding developments and start with stability in classical algebraic K-theory. The underlying philosophy is explained well in [16, Part IV] and in the introduction to [3]. For simplicity let us list the results for the case of a commutative noetherian ring R of Krull dimension d. The set of isomorphism classes of finitely generated projective R-modules of constant rank n we denote by $SK_0(n, R)$. (We choose this notation because the limit for $n\to\infty$ of the $SK_0(n, R)$, with the usual stabilization maps, is what is commonly called $SK_0(R)$.) Similarly we let $K_1(n, R)$, $K_2(n, R)$ denote unstable versions of Bass' K_1 and Milnor's K_2 respectively (cf. [15] or [13]). We have:

$SK_0(n, R)\to SK_0(n+1, R)$ is surjective for $n\geqq d$ (Serre, cf. [25] Théorème 1) and injective for $n\geqq d+1$ (Bass [3], Prop. 10.1).

$K_1(n, R)\to K_1(n+1, R)$ is surjective for $n\geqq d+1$ (Bass [3], Prop. 11.2) and injective for $n\geqq d+2$ (Bass, Vaserstein [29], Theorem 3.2).

$K_2(n, R)\to K_2(n+1, R)$ is surjective for $n\geqq d+2$ (Dennis, Vaserstein [31]) and injective for $n\geqq d+3$ (Suslin and Tulenbayev [26], van der Kallen [13]. In [26] one also finds nice proofs of the two preceding results).

1.4. The facts just listed suggest a stability conjecture of the following form:

$K_m(n, R)\to K_m(n+1, R)$ is surjective for $n\geqq m+d$, injective for $n\geqq m+d+1$. $(m\geqq 1.)$

To give this conjecture a meaning one has to define the $K_m(n, R)$ for all $m\geqq 1$. So one has to choose one of the approaches to higher algebraic K-theory and see what the unstable K-groups $K_m(n, R)$ would be in that approach. If the conjecture is to have a fair chance, then the $K_m(n, R)$ should agree with those of Bass and Milnor for $n=1, 2$ resp. This suggests a K-theory of Volodin type (see [27]). However, one often proceeds differently and considers the unstable K-groups $\pi_m(BGL_n^+(R))$ of Quillen's BGL^+-approach $(n\geqq 3,\ m\geqq 1)$. The $\pi_2(BGL_n^+(R))$ do not always agree with Milnor's $K_2(n, R)$ and the conjecture, as stated above, is false in BGL^+ context, as is shown by the classical examples $R=\mathbb{F}_2$ and $R=\mathbb{Z}$ $(m=2)$. (Compare [13], Sect. 8.) Of course the stable K-group $K_2(R)$ of Quillen is the same as Milnor's and in fact $\pi_2(BGL_n^+(R))$ agrees with Milnor's $K_2(n, R)$ for $n\geqq 5$. (Here we use that R is commutative. See [14] and compare also with [15].) We may sum up the stability behaviour of the $\pi_2(BGL_n^+(R))$ as follows.

$\pi_2(BGL_n^+(R))\to\pi_2(BGL_{n+1}^+(R))$ is surjective for $n\geqq\max(4, d+2)$ and injective for $n\geqq\max(5, d+3)$.

This description of the range of stability is supported by examples of non-stability for many values of d. (See [13], Sect. 7.) At this moment it seems too much to ask for a well motivated conjecture for the range of stability of $\pi_m(BGL_n^+(R))$ for all $m \geqq 1$, giving in a natural way the established range for $m = 2$. What we can offer is the following theorem:

$\pi_m(BGL_n^+(R)) \to \pi_m(BGL_{n+1}^+(R))$ is surjective for $n \geqq 2m + \max(1, d) - 1$ and injective for $n \geqq 2m + \max(1, d) + 1$ $(m \geqq 1, n \geqq 3)$.

For a more general formulation, not restricted to finite dimensional commutative noetherian rings, see Corollary 4.12 below. It is hoped that this stability result will help in extending to higher dimensions Quillen's theorem on finite generation of K-groups of commutative finitely generated regular $\mathbb{Z}$-algebras of dimension one. (See [4], Sect. 9, and [22, 23].)

1.5. By way of a Hurewicz argument stability for the $\pi_m(BGL_n^+(R))$ follows from stability for the $H_m(E_n(R))$ $(m \geqq 2)$. (See 4.12 below.) As a first approximation to stability for the $H_m(E_n(R))$ one may study the simpler problem of stability for the $H_m(GL_n(R))$. Quillen (unpublished) has shown that, when R is a field different from $\mathbb{F}_2$, the map $H_m(GL_n(R)) \to H_m(GL_{n+1}(R))$ is an isomorphism for $n \geqq m+1$. As the present work follows the same general principles, let us sketch Quillen's approach, stressing features that are relevant to us. Suppose G is a group, H a subgroup, and suppose there is a nice sort of geometry associated with the set of right cosets G/H. (For example, when $G = GL_n(k)$ where k is a field, choose a non-zero vector v in k^n and let H be the stabilizer of v in G. The set G/H may be identified with the orbit of v, which is almost the same as k^n, and in this case we may associate with G/H the geometry of linear n-space k^n.) Now construct a simplicial complex T, based on combinatorial properties of the geometry, such that G acts naturally on T and H is the stabilizer of some 0-simplex. When G acts transitively on simplices of fixed dimension, for each dimension, and moreover T is highly connected, one gets a spectral sequence relating the homology of G with the homology of the stabilizers in G of simplices of T. This spectral sequence may be useful in an inductive argument, e.g. when one wants to show that in a certain range the homology of G is the same as the homology of H. (Compare with the following situation which one meets when studying homotopy groups of the Lie groups $SO_n(\mathbb{R})$: There is a fibration $SO_n(\mathbb{R}) \to SO_{n+1}(\mathbb{R}) \to S^n$ and the fact that S^n is $(n-1)$-connected makes that $\pi_i(SO_n(\mathbb{R})) \to \pi_i(SO_{n+1}(\mathbb{R}))$ is an isomorphism for $i \leqq n-2$.)

Quillen tried several simplicial complexes. One is the Tits building, which is known to be highly connected by the Solomon-Tits theorem. Another one was based on unimodular sequences of vectors. Quillen showed it to be highly connected in the case of local rings and he conjectured a similar result for finite dimensional noetherian rings. (See [33], Sect. 1.) The proof of this conjecture is one of the goals of Sect. 2 below.

1.6. The same approach to stability of homology groups has since been followed by J.B. Wagoner, Karen Vogtmann, R. Alperin, Ruth Charney, W.G. Dwyer and H. Maazen. (For some different type of work see [5, 12] and the short survey [6].) Wagoner modified Quillen's treatment of $H_m(GL_n(R))$ when R is local ([33]); Vogtmann dealt with homology of orthogonal groups over a field of characteristic zero ([32]); Alperin discussed homology of the complex unitary groups SU_n([1]). Ruth Charney proved stability theorems for homology of linear groups over Dedekind domains. She showed for instance, when R is a PID, that $H_k(SL_{n+1}(R), SL_n(R); \mathbb{Z})=0$ for $n \geqq 3k$, $H_k(SL_{n+1}(R), SL_n(R); \mathbb{Z}[\frac{1}{2}])=0$ for $n \geqq 2k$ (see [7]). She has since extended her work to the case of $GL_n(\mathbb{Z}[\mathbb{Z}_p])$. To get around certain difficulties encountered by Quillen, Charney invented "split buildings" and proved that they are highly connected. The point of split buildings is that one gets stabilizer subgroups which are much easier to handle than the ones arising from an ordinary Tits building. This was exploited further by Dwyer in his work on homology with twisted coefficients [9]. Maazen, working independently from Charney, found a different solution for the same difficulties. In [18] he deals with all stabilizer subgroups that are encountered, by inventing new simplicial complexes to let them act upon.

That leads again to more types of stabilizers, but the process stops. The simplicial complexes in [18] are all analogous to the space built from unimodular sequences by Quillen. The modifications that are needed reflect the differences between the geometries associated with the different types of groups. Maazen shows that all his stabilizer subgroups have, in a certain range, the same homology with constant coefficients as the full group. This fact doesn't seem to have an analogue with twisted coefficients and it fails for the stabilizers which one meets with split buildings (or with "ordinary" buildings). This explains why in these other contexts one argues with certain relative homology groups rather than with the absolute ones (cf. [32, 7, 9]). We will describe both an inductive procedure for relative groups (with twisted coefficients) and one for absolute homology groups (with constant coefficients). See Sects. 5 and 4 respectively.

1.7. Maazen's main result concerns stability for $H_m(GL_n(R))$ when R is a subring of $\mathbb{Q}$ (cf. [19]). He gets a better range of stability than Charney. To get a good feeling for what the general technique should look like, he started with studying the stability problem for the homology of the symmetric groups. (Recall that it is often profitable to view the symmetric group S_n as $GL_n(F)$, where F is the hypothetical field with one element. More generally the Weyl group of a reductive split algebraic group may be viewed in this way.) From the work of Nakaoka [21] one knows what the range of stability is in the case of symmetric groups. It is most simply described by saying that $H_m(S_n) \to H_m(S_{n+1})$ is an isomorphism for $n \geqq 2m$.

This is indeed what Maazen gets. (The "geometry" in the case of S_n is just a set of n distinct points.) Maazen proceeds by proving stability for $H_m(GL_n(R))$ when R is a field. The range is not as good as in Quillen's result, but the techniques are such that $\mathbb{F}_2$ need not be excluded and such that local rings can be treated, to a large extent, just like fields. (The results in [33] are much messier.) To generalize the arguments further, one needs to prove that the simplicial complexes involved are still highly connected when R is a more general ring. Maazen gets very sharp results of this type when R is a subring of $\mathbb{Q}$, and partial results when R is euclidean. For us his argument in the case of fields is more important however. It is an inductive argument with a combinatorial flavor. Now replace the field by any ring which satisfies one of Bass' stable range conditions. This author found that one can keep the induction going, in a rather unexpected way, by reorganizing it slightly and then inserting a change of co-ordinates of the type that has been standard since the time that Bass introduced his stable range conditions. (An earlier proof was not as tight as the one presented here. It employed techniques from [28] to generalize Maazen's proof in a more straightforward manner. The resulting connectedness statements were only half as good.) This paper is meant as an exposition of the Maazen-Dwyer approach as it looks after our contribution. For simplicity we avoid such notions as "systems of coefficients on a category", although [18] shows that using them can be illuminating.

Under normal circumstances the results which we describe in this paper would have been presented jointly with H. Maazen. However Maazen has left mathematics and is now a student at the Utrecht Divinity School.

§ 2. An Acyclicity Theorem

2.1. In this section we show that certain simplicial complexes are highly connected.

2.2. *Convention.* Throughout this paper R denotes an associative ring with unit, satisfying Bass' stable range condition $SR_{\mathrm{sdim}+2}(R)$, where sdim is a given non-negative integer. In the terminology of [28] this means that we assume $\mathrm{s.r.}(R)<\infty$ and fix an integer sdim with $\mathrm{sdim} \geqq \mathrm{s.r.}(R)-1$.

Standard example. When R is finitely generated as a module over a central subring S and S has a noetherian maximal spectrum of dimension d, Bass' Stable Range Theorem ([2], Chap. V., Thm. 3.5) tells us that we may take $\mathrm{sdim}=d$.

2.3. *Definition.* When V is a set, $\mathcal{O}(V)$ denotes the poset (=partially ordered set) of ordered sequences of distinct elements of V, the length of each sequence being at least one. The partial ordering on $\mathcal{O}(V)$ is defined by refinement: $(v_1, \ldots, v_m) \leqq (w_1, \ldots, w_n)$ if and only if there is a strictly increasing map ϕ: $\{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $v_i = w_{\phi(i)}$. A subset F of $\mathcal{O}(V)$ is said to satisfy the *chain condition* when it contains with any element $(w_1, \ldots, w_n)$ also the $(v_1, \ldots, v_m)$ in $\mathcal{O}(V)$ with $(v_1, \ldots, v_m) \leqq (w_1, \ldots, w_n)$. We always view a subset of a poset as a subposet with the induced partial ordering. When F is a subposet of $\mathcal{O}(V)$ which satisfies the chain condition, F_* denotes (the geometric realization of) the (semi-) simplicial set whose (non-degenerate) d-simplices are the $(v_1, \ldots, v_{d+1})$ that are in F. Recall that the classifying space $|F|$ of F (i.e. the geometric realization of the nerve of the category associated with F) is the barycentric subdivision of the space F_* (cf. [22], p. 81=89).

2.4. Let $R^{(\infty)}$ denote the free right R-module on the countable basis $e_1, e_2, \ldots$ and let R^n denote the submodule generated by $e_1, \ldots, e_n$. Let $\mathcal{U}$ denote the subposet of $\mathcal{O}(R^{(\infty)})$ consisting of unimodular sequences. (Recall that a sequence of vectors $(v_1, \ldots, v_m)$ in a right R-module M is called unimodular when $v_1, \ldots, v_m$ is a basis of a free direct summand of M.) Note that when $(v_1, \ldots, v_m) \in \mathcal{O}(R^t)$, it is the same to say that $(v_1, \ldots, v_m)$ is unimodular as a sequence of vectors in R^t or as a sequence of vectors in $R^{(\infty)}$. We call an element of $\mathscr{U}$ a *frame*. A k-frame is a frame consisting of k vectors. When $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_m)$ are frames, we call them *transversal* if $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is also a frame. When $(v_1, \ldots, v_n)$ is a frame, we write $\mathscr{U}_{(v_1, \ldots, v_n)}$ for the set of frames that are transversal to $(v_1, \ldots, v_n)$. More generally, when $F \subseteq \mathscr{U}$ and $(v_1, \ldots, v_n)$ is a frame, we write $F_{(v_1, \ldots, v_n)}$ for the set of frames $(w_1, \ldots, w_m)$ with $(w_1, \ldots, w_m, v_1, \ldots, v_n) \in F$. Note that

$$(F_{(v_1, \ldots, v_n)})_{(w_1, \ldots, w_m)} = F_{(w_1, \ldots, w_m, v_1, \ldots, v_n)}.$$

2.5. Recall that a space X is (-1)-connected when it is non-empty, 0-connected when it is non-empty and connected, 1-connected when it is non-empty and simply connected. For $k \geqq 1$ it follows from the Hurewicz theorem that X is k-connected if and only if it is 1-connected and the reduced homology groups $\tilde{H}_i(X)$ vanish for $0 \leqq i \leqq k$. For $k < -1$ the condition of k-connectedness is void. Let $F \subseteq \mathcal{O}(V)$ satisfy the chain condition. As usual we call F contractible (k-connected etc.) when $|F|$ has this property or, equivalently, when F_* has this property.

2.6. The main result of this section is:

Theorem. *Let R satisfy Bass' stable range condition $SR_{\mathrm{sdim}+2}(R)$, cf. 2.2. Let δ be 0 or 1.*

(i) *$\mathcal{O}(R^n + e_{n+1}\delta) \cap \mathscr{U}$ is $(n-2-\mathrm{sdim})$-connected.*

(ii) *$\mathcal{O}(R^n + e_{n+1}\delta) \cap \mathscr{U}_{(v_1, \ldots, v_k)}$ is $(n-2-\mathrm{sdim}-k)$-connected for all k-frames $(v_1, \ldots, v_k)$ in $\mathscr{U}$, $k \geqq 1$.*

(iii) *$\mathcal{O}((R^n + e_{n+1}\delta) \cup (R^n + e_{n+1}\delta + e_{n+2})) \cap \mathscr{U}$ is $(n-1-\mathrm{sdim})$-connected.*

(iv) *$\mathcal{O}((R^n + e_{n+1}\delta) \cup (R^n + e_{n+1}\delta + e_{n+2})) \cap \mathscr{U}_{(v_1, \ldots, v_k)}$ is $(n-1-\mathrm{sdim}-k)$-connected for all k-frames $(v_1, \ldots, v_k)$ in $\mathscr{U}$, $k \geqq 1$.*

2.7. *Remarks.* 1. Parts (i) and (ii) will be needed in later sections; parts (iii) and (iv) have been added for the sake of the proof.

2. The theorem and its proof can easily be generalized by means of Vaserstein's theory of "big" modules ([30]).

For any right R-module M one then gets statements about subposets of a poset of "frames in $M \oplus R^{(\infty)}$". We leave the details to the interested reader.

3. The theorem is sometimes sharp. For instance, when $R = \mathbb{F}_2 \times \mathbb{F}_2$ one may take sdim $= 0$ and $\mathcal{O}(R + e_2) \cap \mathscr{U}$ is indeed non-empty, but it is not connected. Also, when R is commutative it is not difficult to see that $\mathcal{O}(R^2) \cap \mathscr{U}$ is connected if and only if $E_2(R)$ acts transitively on unimodular columns of length 2. (i.e. on 1-frames in $\mathcal{O}(R^2)$.) There are many examples of 1-dimensional rings for which this transitivity fails (cf. [8]). On the other hand [18] Chap. III gives examples of d-connectedness with d higher than indicated in the theorem. (Maazen does not worry about homotopy groups but just about homology, because that is all he needs. We also do not need the homotopy groups in later sections, but we wish to confirm the conjecture of Quillen which is mentioned in [33]. Note that in fact we only prove part of this conjecture and have left the rest as an exercise in Remark 2 above.)

4. The estimate of connectivity for $\mathcal{O}(R^n + e_{n+1}) \cap \mathscr{U}$ in part (i) can usually be improved upon, as one sees from:

Proposition. *Let R and d be as in the standard example in 2.2. Assume that $R/\mathrm{Rad}(R)$ has no zero divisors or that R has no finite ring as a homomorphic image* (Rad = *Jacobson radical). Then $\mathcal{O}(R^n+e_{n+1})\cap\mathcal{U}$ is $(n-1-d)$-connected.*

We only give an indication of the proof of this proposition, as we will not need it. In the case that $R/\mathrm{Rad}(R)$ is free of zero divisors the idea is to modify the proof of the theorem, using Theorem 2.6 of [29]. In the case that R has no finite homomorphic image, it is more or less well known how to proceed. One shows, by the method used in the proof of Bass' Stable Range Theorem that the following holds: When X is a finite set of $(k+1)$-frames, $k\geqq 0$, there is a vector v in $R^{d+k+1}+e_{d+k+2}$ such that the 1-frame (v) is transversal to all frames in X. It easily follows from this that any compact subspace of the k-skeleton of $F_*=(\mathcal{O}(R^n+e_{n+1})\cap\mathcal{U})_*$ is contained in a contractible subspace of F_*, when $n\geqq d+k+1$. In fact this general position approach may be used to give a rather direct proof of all of Theorem 2.6. in this case, with sdim $=d$. Thus Quillen's conjecture on the degree of connectivity of $\mathcal{O}(R^n)\cap\mathcal{U}$ in [33] only presents difficulties when R has a finite homomorphic image. (Incidentally, that is how Quillen came to guessing $(n-2-d)$-connectedness.)

2.8. The remainder of Sect. 2 is devoted to the proof of Theorem 2.6. and contains no material that is needed for an understanding of the later sections. Before extracting some technical lemmas from [18] we now introduce some more conventions.

2.9. When $k\geqq -1$, we say that a space X is k-acyclic if it is non-empty and $\tilde{H}_i(X)=0$ for $0\leqq i\leqq k$. When $k<-1$ the condition of k-acyclicity is void.

2.10. When F is a poset, S a subset of F, x an element of F, we write $S^+(x)$ for $\{y\in S\colon y\geqq x\}$ and $S^-(x)$ for $\{y\in S\colon y\leqq x\}$. Further $\mathrm{Link}_S(x)$ denotes the link of x in S, i.e.

$$\mathrm{Link}_S(x)=\mathrm{Link}_S^-(x)\cup\mathrm{Link}_S^+(x)$$

where $\mathrm{Link}_S^-(x)=\{y\in S\colon y<x\}$ and $\mathrm{Link}_S^+(x)=\{y\in S\colon y>x\}$. One has $|\mathrm{Link}_S(x)| =|\mathrm{Link}_S^-(x)|*|\mathrm{Link}_S^+(x)|$, where $*$ denotes the join, as usual.

Recall that the significance of Link is that, when $x\notin S$, the space $|S\cup\{x\}|$ is obtained from $|S|$ by adding a cone over $|\mathrm{Link}_S(x)|$.

2.11. **Lemma.** *Let F be a poset, $S\subseteq F$ such that for each $x\in F$ the poset $S^-(x)$ has a supremum (in itself!) Then S is a deformation retract of F.*

Proof. Define $r\colon F\to S$ by $r(x)=\sup(S^-(x))$. So $r(x)=x$ for $x\in S$.
Let $i\colon S\to F$ denote the inclusion map. Then i, r are morphisms of posets and $i(r(x))\leqq x$ for all $x\in F$, so we have a natural transformation of functors $ir\to \mathrm{id}_F$ and we may apply [24], Proposition 2.1.

Remark. Whenever we will say that a subposet of some poset is a deformation retract, we will be applying this lemma.

2.12. **Lemma.** *Let $F\subseteq\mathcal{U}$ satisfy the chain condition. Let d be an integer and $(v_1,\dots,v_m)\in F$ such that for all $(w_1,\dots,w_n)\in F^+((v_1,\dots,v_m))$ the poset $F_{(w_1,\dots,w_n)}$ is $(d-n)$-acyclic. Then $\mathrm{Link}_F((v_1,\dots,v_m))$ is $(d-1)$-acyclic.*

Proof. Using that for $0 \leqq k \leqq m-2$ there are natural $1-1$ correspondences between k-simplices of $\operatorname{Link}_F^-((v_1, \ldots, v_m))_*$, subsets of size $k+1$ of $\{1, \ldots, m\}$, k-dimensional faces of the standard $(m-1)$-simplex, one easily sees that $|\operatorname{Link}_F^-((v_1, \ldots, v_m))|$ is the barycentric subdivision of the boundary of the standard $(m-1)$-simplex, hence an $(m-2)$-sphere. It follows that

$$\begin{aligned}|\operatorname{Link}_F((v_1, \ldots, v_m))| &= |\operatorname{Link}_F^-((v_1, \ldots, v_m))| * |\operatorname{Link}_F^+((v_1, \ldots, v_m))| \\ &= S^{m-1}\, |\operatorname{Link}_F^+((v_1, \ldots, v_m))|.\end{aligned}$$

Remains to show that $\operatorname{Link}_F^+((v_1, \ldots, v_m))$ is $(d-m)$-acyclic.

We will prove this by induction on m. (Here F and d are allowed to vary.) Put $P_0 = \{(w_1, \ldots, w_n) \in \operatorname{Link}_F^+((v_1, \ldots, v_m)) : w_n = v_m\}$. Put $P_1 = \{(w_1, \ldots, w_r) \in F$: for some n with $1 \leqq n \leqq r$ one has $(w_1, \ldots, w_n) \in P_0\}$. Then P_0 is a deformation retract of P_1. If $m=1$ then P_0 is isomorphic, as a poset, with $F_{(v_1)}$, which is $(d-1)$-acyclic. If $m>1$ then P_0 is isomorphic with $\operatorname{Link}^+_{F_{(v_m)}}((v_1, \ldots, v_{m-1}))$, which is $((d-1)-(m-1))$-acyclic by induction hypothesis. (Use that $(F_{(v_m)})_{(w_1, \ldots, w_n)} = F_{(w_1, \ldots, w_n, v_m)}$ is $(d-1-n)$-acyclic for $(w_1, \ldots, w_n) \in F^+_{(v_m)}((v_1, \ldots, v_{m-1}))$.) So in any case P_1 is $(d-m)$-acyclic. The complement of P_1 in $\operatorname{Link}_F^+((v_1, \ldots, v_m))$ consists of the elements $(v_1, \ldots, v_m, z_1, \ldots, z_q)$ in F with $q \geqq 1$. We will now add these elements to P_1, first those with $q=1$, next those with $q=2$, etcetera. Put $Q_r = P_1 \cup \{(v_1, \ldots, v_m, z_1, \ldots, z_q) \in F : 1 \leqq q \leqq r\}$. We know that $Q_0 = P_1$ is $(d-m)$-acyclic and we want to show that Q_r is $(d-m)$-acyclic for $r \geqq 0$. (As $\operatorname{Link}_F^+((v_1, \ldots, v_m)) = \bigcup_{r \geqq 0} Q_r$, that will prove the lemma.) Because $Q_{r+1} \setminus Q_r$ is discrete, one passes from $|Q_r|$ to $|Q_{r+1}|$ by adding a cone over $\operatorname{Link}_{Q_r}((v_1, \ldots, v_m, z_1, \ldots, z_{r+1}))$ for each $(v_1, \ldots, v_m, z_1, \ldots, z_{r+1})$ in F. As in the beginning of the proof we see that $|\operatorname{Link}_{Q_r}((v_1, \ldots, v_m, z_1, \ldots, z_{r+1}))| = S^r\, |\operatorname{Link}^+_{Q_r}((v_1, \ldots, v_m, z_1, \ldots, z_{r+1}))|$. When $m=1$ the poset $\operatorname{Link}^+_{Q_r}((v_1, z_1, \ldots, z_{r+1}))$ has as a deformation retract the subposet

$$\{(w_1, \ldots, w_s, v_1, z_1, \ldots, z_{r+1}) : (w_1, \ldots, w_s) \in F_{(v_1, z_1, \ldots, z_{r+1})}\}.$$

As this subposet is isomorphic with $F_{(v_1, z_1, \ldots, z_{r+1})}$, it is $(d-r-2)$-acyclic and $\operatorname{Link}_{Q_r}((v_1, z_1, \ldots, z_{r+1}))$ is $(d-2)$-acyclic. Similarly, when $m>1$, $\operatorname{Link}_{Q_r}((v_1, \ldots, v_m, z_1, \ldots, z_{r+1}))$ is $(d-m-1)$-acyclic because $\operatorname{Link}^+_{Q_r}((v_1, \ldots, v_m, z_1, \ldots, z_{r+1}))$ has a deformation retract isomorphic with the poset $\operatorname{Link}^+_{F_{(v_m, z_1, \ldots, z_{r+1})}}((v_1, \ldots, v_{m-1}))$ which is $((d-r-2)-(m-1))$-acyclic by the induction hypothesis. All in all one passes from $|Q_r|$ to $|Q_{r+1}|$ by adding cones over $(d-m-1)$-acyclic links. As Q_0 is $(d-m)$-acyclic it follows from a Mayer-Vietoris sequence that Q_r is $(d-m)$-acyclic, as required.

2.13. **Lemma.** *Let $F \subseteq \mathscr{U}$ satisfy the chain condition. Let $X \subseteq R^{(\infty)}$.*

(i) *Assume that $\mathcal{O}(X) \cap F$ is d-connected and that, for all frames $(v_1, \ldots, v_k)$ in $F \setminus \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1, \ldots, v_k)}$ is $(d-k)$-connected. Then F is d-connected.*

(ii) *Assume that for all frames $(v_1, \ldots, v_k)$ in $F \setminus \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1, \ldots, v_k)}$ is $(d-k+1)$-connected. Assume further that there is a 1-frame (y_0) in F with $\mathcal{O}(X) \cap F \subseteq F_{(y_0)}$. Then F is $(d+1)$-connected.*

Proof. Proof of (i):
Put $P_0 = \{(v_1, \ldots, v_k) \in F$: at least one of the v_i is in $X\}$. Then P_0 has $\mathcal{O}(X) \cap F$ as a deformation retract and is therefore d-connected. Put $P_q = P_0 \cup \{(w_1, \ldots, w_r) \in F : r \leqq q\}$. We consider $\mathrm{Link}_{P_q}((w_1, \ldots, w_{q+1}))$ for some $(w_1, \ldots, w_{q+1}) \in P_{q+1} \setminus P_q$. Put $Q = \mathcal{O}(X \cup \{w_1, \ldots, w_{q+1}\}) \cap F$. Then $\mathrm{Link}_{P_q}((w_1, \ldots, w_{q+1}))$ has $\mathrm{Link}_Q((w_1, \ldots, w_{q+1}))$ as a deformation retract and this deformation retract is $(d-1)$-acyclic by the previous lemma.

So one passes from $|P_q|$ to $|P_{q+1}|$ by adding cones over $(d-1)$-acyclic links. As in the previous proof we find that P_q is d-acyclic for all $q \geqq 0$, and, when $d \geqq 1$, the Van Kampen Theorem shows that P_q is also simply connected. Now note that $F = \bigcup_{q \geqq 0} P_q$.

Proof of (ii): Note that $\mathcal{O}(X) \cap F = \mathcal{O}(X) \cap F_{(y_0)}$ is d-connected. (Clearly $y_0 \notin X$.) Define P_q as in the proof of (i). Then P_0 is d-connected again. But now $\mathrm{Link}_{P_q}((w_1, \ldots, w_{q+1}))$ is d-acyclic for $(w_1, \ldots, w_{q+1}) \in P_{q+1} \setminus P_q$. In particular $\mathrm{Link}_{P_0}((y_0))$ is d-acyclic. Define $\psi: \mathcal{O}(X) \cap F \to \mathrm{Link}_{P_0}((y_0))$ by $\psi((v_1, \ldots, v_k)) = (v_1, \ldots, v_k, y_0)$. The image $\mathrm{Im}\,\psi$ of ψ is isomorphic with $\mathcal{O}(X) \cap F$ and $\mathcal{O}(X) \cap F$ is a deformation retract of P_0 via a retraction whose restriction to $\mathrm{Im}\,\psi$ is inverse to ψ. It follows that $\mathrm{Im}\,\psi$ is a subposet of $\mathrm{Link}_{P_0}((y_0))$, homotopy equivalent with P_0 via the inclusion map. Therefore $\tilde{H}_i(|\mathrm{Link}_{P_0}((y_0))|) \to \tilde{H}_i(|P_0|)$ is surjective for $i \geqq 0$ and one sees from a Mayer-Vietoris sequence that $P_0 \cup \{(y_0)\}$ is $(d+1)$-acyclic. Also, when $d \geqq 0$, it follows from Van Kampen's Theorem that $P_0 \cup \{(y_0)\}$ is simply connected. So $P_0 \cup \{(y_0)\}$ is $(d+1)$-connected. To pass from $|P_0 \cup \{(y_0)\}|$ to $|P_1|$, or from $|P_q|$ to $|P_{q+1}|$ when $q \geqq 1$, one adds cones over d-acyclic links. From a Mayer-Vietoris sequence (and, when $d \geqq 0$, the Van Kampen Theorem) one sees that the P_q are $(d+1)$-connected for $q \geqq 1$, so that $\mathcal{O}(X) \cap F$ is an increasing union of $(d+1)$-connected posets, hence $(d+1)$-connected.

2.14. The proof of Theorem 2.6 will now proceed inductively in the following way. For each of the posets in the theorem we either prove d-connectedness directly (when d is small) or we reduce it to sufficient connectedness of smaller posets by means of Lemma 2.13. To be on the safe side, let us introduce a parameter "size" by saying that the poset in part (i) of the theorem has size $2n$, the poset in part (ii) has size $2n-k$, the poset in part (iii) has size $2n+1$, the poset in part (iv) has size $2n-k+1$. When discussing one of these posets our induction hypothesis will be that the theorem is correct for all posets of strictly smaller size. It is easy to see that the theorem is correct for posets of negative size. Let us now discuss the four cases in the theorem consecutively.

2.15. Case (i). Let $F = \mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}$, $d = n - 2 - \mathrm{sdim}$. We have to show that F is d-connected. When $n = 0$ this is obvious. When $n > 0$, choose $X = (R^{n-1} + e_{n+1}\delta) \cup (R^{n-1} + e_n + e_{n+1}\delta)$. Note that $\mathcal{O}(X) \cap F$ is d-connected by the induction hypothesis, because X is the same as $(R^{n-1} + e_n\delta) \cup (R^{n-1} + e_n\delta + e_{n+1})$, up to a change of co-ordinates. In the same way one checks that all conditions of part (i) of Lemma 2.13 are satisfied.

2.16. Case (ii). This case is similar to the previous one.

2.17. Case (iii). Let $F=\mathcal{O}((R^n+e_{n+1}\delta)\cup(R^n+e_{n+1}\delta+e_{n+2}))\cap\mathcal{U}$, $d=n-2-\text{sdim}$. We have to show that F is $(d+1)$-connected. Choose $X=R^n+e_{n+1}\delta$, $y_0=e_{n+1}\delta+e_{n+2}$. Consider $\mathcal{O}(X)\cap F_{(v_1,\ldots,v_k)}$ for some $(v_1,\ldots,v_k)\in F\setminus\mathcal{O}(X)$. Say $v_1\notin X$. (Otherwise permute the v_i). The $(n+2)$-nd co-ordinate of v_1 equals 1. For $\alpha_i\in R$, $v_i'=v_i-v_1\alpha_i$, we have $\mathcal{U}_{(v_1,\ldots,v_k)}=\mathcal{U}_{(v_1,v_2',\ldots,v_k')}$ by an exercise in linear algebra. Choose α_i such that the $(n+2)$-nd co-ordinate of v_i' vanishes for $2\leqq i\leqq k$. Then $\mathcal{O}(X)\cap F_{(v_1,\ldots,v_k)}=\mathcal{O}(X)\cap\mathcal{U}_{(v_2',\ldots,v_k')}$, again by an exercise in linear algebra. By the induction hypothesis the poset $\mathcal{O}(X)\cap\mathcal{U}_{(v_2',\ldots,v_k')}$ is $(d-k+1)$-connected and we may apply part (ii) of Lemma 2.13.

2.18. So far the arguments are essentially the same as in [18]. Now we are going to apply the stable range condition $SR_{\text{sdim}+2}(R)$. Case (iv). Let $F=\mathcal{O}(Y)\cap\mathcal{U}_{(v_1,\ldots,v_k)}$, where $Y=(R^n+e_{n+1}\delta)\cup(R^n+e_{n+1}\delta+e_{n+2})$. Put $d=n-1-\text{sdim}-k$. We have to show that F is d-connected. When $n\leqq\text{sdim}$ there is nothing to prove. When $n=\text{sdim}+1$ the only interesting case is that $k=1$. Then we need to show that F is non-empty. Now a standard application of $SR_{\text{sdim}+2}$ (cf. [29], Thm. 2.3) tells us that there is $g\in GL(R^{(\infty)})$ such that $g(Y)=Y$ and such that $(g(v_1), e_{n+2})$, is a 2-frame. Changing co-ordinates according to g we arrive at the situation that $(e_{n+2})\in F$. Remains the case $n\geqq\text{sdim}+2$. Again we change co-ordinates. Now we do it so that the first co-ordinate of v_1 becomes equal to 1, again without changing Y. (This is also a standard application of $SR_{\text{sdim}+2}$.) After the change we put $X=\{v\in Y$: the first co-ordinate of v vanishes$\}$. By a computation as in the earlier cases we see that part (i) of Lemma 2.13 applies.

§ 3. Some Spectral Sequences

3.1. In this section we introduce some groups, let them act on simplicial complexes that have been studied in Sect. 2, and obtain spectral sequences for homology groups.

3.2. *Conventions.* Let $XL(R)$ denote a subgroup of the stable general linear group $GL(R)$, containing the elementary subgroup $E(R)$. We have three examples in mind: $XL(R)$ may be $GL(R)$, $E(R)$, $SL(R)$. (In the last case assume R commutative.) As usual $GL_n(R)$ is viewed as a subgroup of $GL(R)$. (cf. [2], Chap. V). Let P, Q be sets of integers with $P\cup Q=[1,t]$ for some $t\geqq 0$, and let $n\geqq t$. (By $[a,b]$ we always mean $\{i\in\mathbb{Z}: a\leqq i\leqq b\}$.) We write G_n^{PQ} for the subgroup of $GL_n(R)\cap XL(R)$ consisting of the elements that fix the column vectors e_j with $j\in P$ and the row vectors e_i^* with $i\in Q$. Here e_i^* denotes the transpose of e_i, hence an element of the dual basis (in the space of row vectors) of the basis $e_1, e_2\ldots$. For instance, when $P=\{1,2\}$, $Q=\{1,3\}$ (hence $t=3$) and $n=5$, then G_n^{PQ} consists of the matrices in $GL_5(R)\cap XL(R)$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$$

We have an "upper inclusion" map $I: G_n^{PQ} \to G_{n+1}^{PQ}$, compatible with the usual stabilization map $GL_n(R) \to GL_{n+1}(R)$. Whenever we have a group and a subgroup, "inc" will stand for the inclusion map. Note that we view G_n^{PQ}, G_{n+1}^{PQ} as subgroups of $GL(R)$ in such a way that $I: G_n^{PQ} \to G_{n+1}^{PQ}$ and inc: $G_n^{PQ} \to G_{n+1}^{PQ}$ stand for the same map. When $P=Q=[1,t]$, we call G_n^{PQ} a *square* group of *size* $n-t$. Such a group is isomorphic with $GL_{n-t}(R) \cap XL(R)$, but, when $t \neq 0$, it is not identical with $GL_{n-t}(R) \cap XL(R)$. (In Sect. 5 the distinction between different isomorphic subgroups of $XL(R)$ will be important, but in Sect. 4 it will hardly matter.) We write $Sq(G_n^{PQ})$ for the largest square group contained in G_n^{PQ}. (So $Sq(G_n^{PQ}) = G_n^{TT}$ where $T = P \cup Q$.)

3.3. For fixed n let $\mathscr{F}_n$ denote the family of groups G_n^{PQ}. It is a finite set, partially ordered by containment. If G, H are in $\mathscr{F}_n$ then H is called a *predecessor* of G when H is maximal among the elements of $\mathscr{F}_n$ that are strictly contained in G. We say that it is a *good* predecessor when H contains the largest square element K of $\mathscr{F}_n$ that is properly contained in G. Thus one can follow a descending route from G to K by taking good predecessors repeatedly. Via such routes we will compare each group in $\mathscr{F}_n$ with a square one and the square ones amongst each other.

3.4. Say $G = G_n^{PQ}$ and H is a good predecessor. Let t be the size of $P \cup Q$ again and put $d = n - t - 2 - \mathrm{sdim}$. Put $\varepsilon = 0$ when G is square and put $\varepsilon = 1$ otherwise. We claim that Theorem 2.6 provides us with a d-connected simplicial complex F_* on which G acts, such that the following (and more) holds. For $0 \leqq k \leqq \max(0, d+\varepsilon+1)$ there is a non-degenerate k-simplex σ_k in F_* so that $\mathrm{Stab}_G(\sigma_k)$, the stabilizer of σ_k in G, is an element of $\mathscr{F}_n$. Moreover the stabilizer of σ_0 is H. Further, G acts transitively on non-degenerate k-simplices of F_* for $0 \leqq k \leqq d+\varepsilon$. (Given G and H this condition of transitivity will make the proper choice of F_* rather obvious.)

We could make a similar claim when H is an arbitrary predecessor of G, but in the sequel we will not even need to consider all good predecessors of G. It will suffice to have for each $G \in \mathscr{F}_n (G \neq \{1\})$ a natural choice of one good predecessor H such that F_* exists. The case $G_n^{PQ} = \{1\}$ will often tacitly be excluded.

3.5. *Examples*. 1. Let $P=Q=[1,t]$, $n > t \geqq 0$. Then G_n^{PQ} acts naturally from the left on $X = e_{t+1}R + \ldots + e_n R$. We may view X as a copy of linear $(n-t)$-space over R on which G_n^{PQ} acts as a group of linear transformations. Take $F_* = (\mathscr{O}(X) \cap \mathscr{U})_*$, $\sigma_k = (e_{t+1}, \ldots, e_{t+k+1})$. (Compare [33].) Then F_* is d-connected by Theorem 2.6(i). (Take $\delta = 0$.) The stabilizer of σ_k is $G_n^{P(k),Q}$ with $P(k) = P \cup [t+1, t+k+1] = [1, t+k+1]$. The stabilizer of σ_0 is a good predecessor of G_n^{PQ}. It is a well-known consequence of $SR_{\mathrm{sdim}+2}(R)$ that G_n^{PQ} acts transitively on non-degenerate k-simplices of F_* for $0 \leqq k \leqq d$. (cf. [29], Theorem 2.3.)

2. Let $P = [1, t-1]$, $Q = [1, t]$, $n \geqq t \geqq 1$. There is a natural transitive action from the left of G_n^{PQ} on $X = e_t + e_{t+1}R + \ldots e_n R$. We may view X as a copy of affine $(n-t)$-space over R on which G_n^{PQ} acts as a group of affine transformations. Take $F_* = (\mathscr{O}(X) \cap \mathscr{U})_*$, $\sigma_k = (e_t, e_t + e_{t+1}, \ldots, e_t + e_{t+k})$. Then F_* is d-connected by Theorem 2.6(i). (Take $\delta = 1$.) The stabilizer of σ_k is $G_n^{P(k),Q}$ with $P(k) = P \cup [t, t+k] = [1, t+k]$. Again the stabilizer of σ_0 is a good predecessor. It

is a well-known consequence of $SR_{\mathrm{sdim}+2}(R)$ that G_n^{PQ} acts transitively on non-degenerate k-simplices of F_* for $0 \leqq k \leqq d+1$.

3. Let $P=[1,t]$, $Q=[1,s]$, $n \geqq t > s \geqq 0$. There is a natural action from the right of G_n^{PQ} on the set of row vectors $X = e_t^* + \mathrm{R}e_{t+1}^* + \ldots + \mathrm{R}e_n^* = e_t^* + e_{t+1}^* R^{\mathrm{op}} + \ldots + e_n^* R^{\mathrm{op}}$. Here R^{op} denotes the opposite ring of R and the free left R-module of row vectors over R is viewed as a free right R^{op}-module $(R^{\mathrm{op}})^{(\infty)}$ in the usual way. Let $F_* = (\mathcal{O}(X) \cap \mathcal{U}^{\mathrm{op}})_*$, where $\mathcal{U}^{\mathrm{op}}$ denotes the poset of unimodular sequences in $(R^{\mathrm{op}})^{(\infty)}$. As $SR_{\mathrm{sdim}+2}(R)$ implies $SR_{\mathrm{sdim}+2}(R^{\mathrm{op}})$ (see [28], Thm. 2), it follows from Theorem 2.6(i) that F_* is d-connected. Take $\sigma_k = (e_t^*, e_t^* + e_{t+1}^*, \ldots, e_t^* + e_{t+k}^*)$. The stabilizer of σ_k in G_n^{PQ} is $G_n^{P, Q(k)}$ with $Q(k) = Q \cup [t, t+k]$. The stabilizer of σ_0 is a good predecessor of G_n^{PQ} and it follows from $SR_{\mathrm{sdim}+2}(R^{\mathrm{op}})$ that G_n^{PQ} acts transitively on non-degenerate k-simplices of F_* for $0 \leqq k \leqq d+1$.

4. Let $P=[1, s-1]$, $Q=[p,t]$, $1 \leqq p \leqq s \leqq t \leqq n$. This generalizes example 2. Let X denote the orbit of e_t under the action (from the left) of G_n^{PQ} on R^n. (One easily checks that $X = e_p + V$ where V is the R-module generated by the e_i with $i \in [1, p-1] \cup [t+1, n]$.) When $p=1$, put $F_* = (\mathcal{O}(X) \cap \mathcal{U})_*$. When $p>1$, put $F_* = (\mathcal{O}(X) \cap \mathcal{U}_{(e_1, \ldots, e_{p-1})})_*$. In either case take $\sigma_k = (e_t, e_t + e_{t+1}, \ldots, e_t + e_{t+k})$. As in Example 2 the space F_* is d-connected, G_n^{PQ} acts on F_*, and the stabilizer of σ_k is $G_n^{P(k), Q}$ with $P(k) = P \cup [t, t+k]$. Again G_n^{PQ} acts transitively on non-degenerate k-simplices for $0 \leqq k \leqq d+1$ and the stabilizer of σ_0 is a good predecessor.

3.6. In general, when $P \cup Q = [1,t]$, there is a permutation π of $[1,t]$ such that the pair $\pi(P)$, $\pi(Q)$ occurs among the four examples in 3.5. For each (P,Q) fix such a permutation π and use it to transfer the construction from the relevant example to G_n^{PQ}. When this leads to an action from the left on a simplicial complex F_*, insert the anti-homomorphism $x \mapsto x^{-1}$ to make it into an action from the right. (This doesn't spoil anything; e.g. the stabilizers are not changed.) Then we have for all $G \in \mathscr{F}_n$, $G \neq \{1\}$, an action from the right on a simplicial complex F_* and further simplices σ_k so that, among other things, $\mathrm{Stab}_G(\sigma_0)$ is a good predecessor of G (cf. 3.4). By choosing the permutation π to be independent of n we arrange that the constructions are compatible with the stabilization maps $I: G_n^{PQ} \to G_{n+1}^{PQ}$.

3.7. Let P, Q, t be as in 3.6. and let $\{1\} \neq G = G_n^{PQ}$. Further let M be a left G-module. Put $d = n - t - 2 - \mathrm{sdim}$ again and recall that $\varepsilon = 0$ when G is square, $\varepsilon = 1$ otherwise. We choose F_*, σ_k as indicated in 3.6.

Proposition. *There is a first quadrant homology spectral sequence* $E(P,Q;n) = E_{rs}^1$, *converging to zero, with*

$$E_{0s}^1 = H_s(G, M) \quad \text{for } s \geqq 0,$$

$$E_{rs}^1 = H_s(\mathrm{Stab}_G(\sigma_{r-1}), M) \quad \text{for } s \geqq 0, 1 \leqq r \leqq d + \varepsilon + 1.$$

For $s \geqq 0$, $d + \varepsilon \geqq 0$, *the differential* $E_{1s}^1 \to E_{0s}^1$ *is the natural map from* $H_s(\mathrm{Stab}_G(\sigma_0), M)$ *to* $H_s(G, M)$.

Proof. Rather than referring to [33], we give such a proof that it will be straightforward to identify the differentials $E_{r+1,s}^1 \to E_{r,s}^1$ for $r \geqq 1$. (See Sect. 4 below.) Let C_k denote the free $\mathbb{Z}$-module on a basis consisting of non-degenerate k-simplices of F_*. As F_* is d-connected there is an exact sequence of right $\mathbb{Z}[G]$-

modules $0 \leftarrow \mathbb{Z} \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_{d+1} \leftarrow Z_{d+1} \leftarrow 0$. Rewrite it as $0 \leftarrow L_0 \leftarrow L_1 \leftarrow \ldots$. (So $L_0 = \mathbb{Z}$, $L_1 = C_0$, etc. When $d \leqq -2$, the Proposition is not very interesting; we then take $L_1 = \mathbb{Z}$, $L_i = 0$ for $i > 1$.) Let $R_* \to M$ be a resolution of M by free (left) $\mathbb{Z}[G]$-modules. The bicomplex of abelian groups

$$\begin{array}{ccccc} & & \vdots & & \vdots & \\ & & \downarrow & & \downarrow & \\ 0 & \leftarrow & E^0_{01} & \leftarrow & E^0_{11} & \leftarrow \ldots \\ & & \downarrow & & \downarrow & \\ 0 & \leftarrow & E^0_{00} & \leftarrow & E^0_{10} & \leftarrow \ldots \\ & & \downarrow & & \downarrow & \\ & & 0 & & 0 & \end{array}$$

with $E^0_{rs} = L_r \underset{\mathbb{Z}[G]}{\otimes} R_s$ is exact in the horizontal direction, so that the homology of its total complex vanishes. (We follow Grothendieck [10], or [11], in that we use spectral sequences for bicomplexes whose horizontal differentials commute with the vertical ones. So the total complex has a differential with signs in it, but the bicomplex E^0_{**} itself is not contaminated by signs.) Taking homology of E^0_{**} first in vertical direction yields a spectral sequence E^1_{rs} converging to zero. We now apply Shapiro's Lemma: The homomorphism $\mathbb{Z} \to C_k$ which sends 1 to σ_k induces a chain map f_* from

$$\mathbb{Z} \underset{\mathbb{Z}[\mathrm{Stab}_G(\sigma_k)]}{\otimes} R_* \quad \text{to} \quad C_k \underset{\mathbb{Z}[G]}{\otimes} R_*.$$

For $0 \leqq k \leqq d + \varepsilon$ the group G acts transitively on the set of non-degenerate k-simplices so that, by a little exercise, the chain map f_* is an isomorphism.

As the complex R_* may be viewed as a resolution of M by free $\mathbb{Z}[\mathrm{Stab}_G(\sigma_k)]$-modules, f_* induces isomorphisms $H_s(\mathrm{Stab}_G(\sigma_k), M) \to E^1_{k+1,s}$, for such k. We use these isomorphisms as identifications. The identification of E^1_{0s} with $H_s(G, M)$ is straightforward and the last statement in the Proposition is also clear.

3.8. The remainder of Sect. 3 will not be used in Sect. 4. As in [33] we now wish to give a relative version of the Proposition. First we recall some homological machinery (cf. [9]). Let G, G' be groups, $\phi: G \to G'$ a homomorphism, M a left G-module, M' a left G'-module. A map $f: M \to M'$ is called *ϕ-linear* when it is $\mathbb{Z}$-linear and $f(gm) = \phi(g) f(m)$ for all $g \in G$, $m \in M$. We will need (doubly) relative groups $H_i(G', G; M', M; \phi, f)$ which fit into a long exact sequence

$$(*) \qquad \ldots H_1(G', M') \to H_1(G', G; M', M; \phi, f) \to H_0(G, M) \to H_0(G', M') \\ \to H_0(G', G; M', M; \phi, f) \to 0,$$

and have suitable functorial properties. (The notation $H_i(G', G; M', M; \phi, f)$ is rather unwieldy. We will use simplified forms such as $H_i(G', G)$ or $H_i(G', G; M', M)$, but not $H_i(\phi, f)$, because the latter is reserved for the map $H_i(G, M) \to H_i(G', M')$.)

3.9. To define the relative homology groups, let us introduce some further notations. **Rep** denotes the category of pairs (G, M) where G is a group and M a left G-module. A morphism $(G, M) \to (G', M')$ in **Rep** is a pair (ϕ, f) with $\phi: G \to G'$ a homomorphism and $f: M \to M'$ a ϕ-linear map.

RelRep is the category whose objects are the morphisms of **Rep**. (A morphism in **RelRep** from (ϕ, f) to (ψ, g) is a pair of morphisms (ρ, h), (σ, k) in **Rep** so that $(\sigma, k)(\phi, f) = (\psi, g)(\rho, h)$.) Further **Ab** denotes the category of abelian groups.

3.10. When G is a group, consider $\mathbb{Z}$ as a right G-module with trivial action. Recall that the $H_i(G, M)$ are often computed by means of a standard resolution ("bar resolution") $B_*(G) \to \mathbb{Z}$ of $\mathbb{Z}$, with $B_*(G)$ depending functorally on G. (Take your favorite version.) For a left G-module M the $H_i(G, M)$ are then the homology groups of the complex $B_*(G) \underset{\mathbb{Z}[G]}{\otimes} M$ and this complex is functorial in (G, M). When $(\phi, f): (G, M) \to (G', M')$ in **Rep**, we define the $H_i(G', G; M', M; \phi, f)$ to be the homology groups of the mapping cone (see [20], p. 46) of the chain map $B_*(G) \underset{\mathbb{Z}[G]}{\otimes} M \to B_*(G') \underset{\mathbb{Z}[G']}{\otimes} M'$. It is clear that these relative groups fit into the long exact sequence (*) of 3.8. The relative H_i are functors from **RelRep** to **Ab**.

3.11. Standard arguments show:

Lemma (cf. [9]). *Let $\phi: G \to G'$ be a homomorphism of groups and*

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & M_2 & \to & M_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M'_1 & \to & M'_2 & \to & M'_3 & \to & 0 \end{array}$$

a commutative diagram with the upper sequence an exact sequence of G-modules, the lower sequence an exact sequence of G'-modules, the vertical maps ϕ-linear. Then there is a long exact sequence

$$\begin{aligned} \ldots H_{i+1}(G', G; M'_3, M_3) &\xrightarrow{\partial} H_i(G', G; M'_1, M_1) \\ &\to H_i(G', G; M'_2, M_2) \to H_i(G', G; M'_3, M_3) \ldots \end{aligned}$$

3.12. We now describe a more flexible way to compute with relative homology groups. Let $(\phi, f): (G, M) \to (G', M')$ again. Choose a projective resolution $P_* \xrightarrow{\varepsilon} M$ and a projective resolution $P'_* \xrightarrow{\varepsilon'} M'$. (So P_* consists of projective left $\mathbb{Z}[G]$-modules, P'_* of projective left $\mathbb{Z}[G']$-modules). There is at least one ϕ-linear chain map $t_*: P_* \to P'_*$, compatible with f. For such a t_* we wish to identify the homology groups of the mapping cone of $1 \otimes t_*: \mathbb{Z} \underset{\mathbb{Z}[G]}{\otimes} P \to \mathbb{Z} \underset{\mathbb{Z}[G']}{\otimes} P'_*$ with the $H_i(G', G; M', M)$. To this end we consider the total complex $T_* = \mathrm{Tot}(B_*(G) \underset{\mathbb{Z}[G]}{\otimes} P_*)$ of the double complex $B_*(G) \underset{\mathbb{Z}[G]}{\otimes} P_*$. (Compare [20], Chap. V, Sect. 9, where Tot is suppressed in the notation.) The augmentation $\varepsilon: P_* \to M$ induces a chain map $T_* \to B_*(G) \underset{\mathbb{Z}[G]}{\otimes} M$ which induces isomorphisms in homology (cf. [20], Chap. V, Thm. 9.3.). Similarly $\varepsilon'\colon P'_* \to M'$ induces a chain map from $T'_* = \mathrm{Tot}(B_*(G') \underset{\mathbb{Z}[G']}{\otimes} P'_*)$ to $B_*(G') \underset{\mathbb{Z}[G']}{\otimes} M'$. Together these two chain maps yield a chain map from the mapping cone of $T_* \to T'_*$ to the mapping cone of $B_*(G) \otimes M \to B_*(G') \otimes M'$, inducing isomorphisms in homology. (Compare the long exact sequences for the two mapping cones and apply the Five Lemma.) We use these isomorphisms as identifications. In the same way we get from the augmentations $B_*(G) \to \mathbb{Z}$ and $B_*(G') \to \mathbb{Z}$ a chain map from the mapping cone of $T_* \to T'_*$ to the mapping cone of $\mathbb{Z} \otimes P_* \to \mathbb{Z} \otimes P'_*$, again inducing isomorphisms in homology. Composing the isomorphisms we get the desired identifications.

Remark. It is not allowed to replace the chain map $T_* \to \mathbb{Z} \otimes P_*$ by one homotopic to it. Such a variation could easily lead to different identification maps. That is why we have been so explicit.

3.13. Let

$$\begin{array}{ccc} (G, M) & \xrightarrow{(\psi, g)} & (K, N) \\ \big\downarrow {\scriptstyle (\varphi, f)} & & \big\downarrow {\scriptstyle (\chi, h)} \\ (G', M') & \xrightarrow{(\psi', g')} & (K', N') \end{array}$$

be a commutative square in **Rep**, representing a morphism $(\varphi, f) \to (\chi, h)$ in **RelRep**. Suppose one has a commutative square of projective resolutions

$$\begin{array}{ccc} P_* & \longrightarrow & Q_* \\ \big\downarrow & & \big\downarrow \\ P'_* & \longrightarrow & Q'_* \end{array}$$

compatible with these data. (So P_* is a $\mathbb{Z}[G]$-projective resolution of M, $P_* \to Q_*$ is a ψ-linear chain map, etc.) The horizontal chain maps induce a chain map from the mapping cone of $\mathbb{Z} \otimes P_* \to \mathbb{Z} \otimes P'_*$ to the mapping cone of $\mathbb{Z} \otimes Q_* \to \mathbb{Z} \otimes Q'_*$, hence they induce maps $q_i\colon H_i(G', G;\ M', M) \to H_i(K', K;\ N', N)$. As the relative H_i are functors $\mathbf{RelRep} \to \mathbf{Ab}$, one likes the q_i to agree with the functorial maps. Indeed a diagram chase shows:

Lemma. *Under the identifications of* 3.12, *the q_i agree with the image under the relative H_i functor of the morphism*

$$((\psi, g), (\psi', g'))\colon (\phi, f) \to (\chi, h).$$

Remark. In the same vein one sees that Dwyer's description of relative homology agrees with ours. So the definition of relative homology groups by means of standard resolutions only serves as a convenient unambiguous reference point.

3.14. In the situation of 3.7. put $G' = G^{PQ}_{n+1}$ and let M' be a left G'-module. Recall that $I\colon G^{PQ}_n \to G^{PQ}_{n+1}$ is the stabilization map. Let $f\colon M \to M'$ be I-linear and let F'_*

denote the analogue of F_* for G^{PQ}_{n+1} instead of G^{PQ}_n. So F'_* is $(d+1)$-connected. Note that our constructions have been chosen so that F'_* contains F_*.

In particular, σ_k is also a k-simplex of F'_* and the action of G' on F'_* leads to a subgroup $\mathrm{Stab}_{G'}(\sigma_k)\in\mathscr{F}_{n+1}$ containing $I(\mathrm{Stab}_G(\sigma_k))$. Let L'_*, R'_* denote analogues for F'_*, M' of the complexes L_*, R_* in the proof of 3.7, but with d not replaced by $d+1$, i.e. with L'_* ending with $(d+1)$-cycles, just like L_*. There is a natural chain map $L_*\to L'_*$ and we may also choose a chain map $R_*\to R'_*$, compatible with f, in an I-linear way. Together these chain maps yield a map of double complexes $L_* \underset{\mathbb{Z}[G]}{\otimes} R_* \to L'_* \underset{\mathbb{Z}[G']}{\otimes} R'_*$. Taking the mapping cone of $L_p \underset{\mathbb{Z}[G]}{\otimes} R_* \to L'_p \underset{\mathbb{Z}[G']}{\otimes} R'_*$ for each p we get a double complex E^0_{**} which is exact in the horizontal direction. Now take homology first in vertical direction. From the proof of 3.7 we see (cf. 3.12, 3.13):

Proposition. *There is a first quadrant homology spectral sequence* $E(P, Q; n+1, n) = E^1_{pq}$, *converging to zero, with*

$$E^1_{0s}=H_s(G', G; M', M) \qquad \text{for } s\geqq 0,$$

$$E^1_{rs}=H_s(\mathrm{Stab}_{G'}(\sigma_{r-1}), \mathrm{Stab}_G(\sigma_{r-1}); M', M) \qquad \text{for } s\geqq 0,$$
$$1\leqq r\leqq d+\varepsilon+1.$$

For $s\geqq 0$ *the differential* $E^1_{1s}\to E^1_{0s}$ *is the natural map from*

$$H_s(\mathrm{Stab}_{G'}(\sigma_0), \mathrm{Stab}_G(\sigma_0); M', M) \quad \textit{to} \quad H_s(G', G; M', M).$$

3.15. *Remark.* Using the $(d+1)$-connectedness of F'_* and the transitivity of the G' action on $(d+\varepsilon+1)$-simplices of F'_*, one checks that for $r=d+\varepsilon+2$ there is a surjective map from $H_0(\mathrm{Stab}_{G'}(\sigma_{r-1}), \mathrm{Stab}_G(\sigma_{r-1}); M', M)$ to E^1_{r0}.

§ 4. Stability for Constant Coefficients

4.1. In this section we analyse the spectral sequences of Proposition 3.7. in the fashion of [18], to find that $H_m(G^{PQ}_n)$ is isomorphic with $H_m(XL(R))$ in a certain range. (Notations as in 3.2.) As a corollary we get stability for the $\pi_m(BGL^+_n(R))$.

4.2. Let us return to the situation of 3.7. and look at the differentials $E^1_{r+1,s}\to E^1_{r,s}$. We have seen already that the differential $E^1_{1s}\to E^1_{0s}$ is (identified with) the natural map from $H_s(\mathrm{Stab}_G(\sigma_0), M)$ to $H_s(G, M)$ for $s\geqq 0$. Now let $1\leqq r\leqq d+\varepsilon$. The differential $E^1_{r+1,s}\to E^1_{r,s}$ is induced by the map $L_{r+1}\to L_r$ which is the map $\sum_{i=1}^{r+1}(-1)^{i+1}\partial_i\colon C_r\to C_{r-1}$, where ∂_i deletes the i-th vector from an $(r+1)$-frame. Choose $m_{i_r}\in G$ so that $\partial_i(\sigma_r)=\sigma_{r-1}m_{i_r}$. (This is possible because G acts transitively on $(r-1)$-simplices.) For each left G-module we denote the map $n\mapsto m_{i_r}n$ by $l(m_{i_r})$. It is an $\mathrm{Int}(m_{i_r})$-linear map, where $\mathrm{Int}(m_{i_r})$ is the homomorphism $g\mapsto m_{i_r}gm_{i_r}^{-1}$. (By abuse of notation we will denote restrictions of $\mathrm{Int}(m_{i_r})$ also by $\mathrm{Int}(m_{i_r})$.)

Inspecting the identifications made in the proof of 3.7., we see that the chain map $E^0_{r+1,*} \to E^0_{r,*}$ which ∂_i induces is (identified with) the chain map which one would use to compute

$$H_s(\mathrm{Int}(m_{ir}), l(m_{ir})) \colon H_s(\mathrm{Stab}_G(\sigma_r), M) \to H_s(\mathrm{Stab}_G(\sigma_{r-1}), M)$$

with, viz. the chain map $\mathbb{Z} \underset{\mathbb{Z}[\mathrm{Stab}(\sigma_r)]}{\otimes} R_* \to \mathbb{Z} \underset{\mathbb{Z}[\mathrm{Stab}(\sigma_{r-1})]}{\otimes} R_*$ induced by the chain map $l(m_{ir})\colon R_* \to R_*$. It follows that the differential $E^1_{r+1,s} \to E^1_{r,s}$ may be viewed as the map $\sum_{i=1}^{r+1} (-1)^{i+1} H_s(\mathrm{Int}(m_{ir}), l(m_{ir}))$. To make this formula correct for $r=0$, simply put $m_{10} = 1 \in G$. When G is square, so that $\varepsilon = 0$, the map

$$H_s(\mathrm{Stab}_G(\sigma_{d+1}), M) \to E^1_{d+2,s}$$

is not always an isomorphism, but it is clear from the above that its composition with the differential $E^1_{d+2,s} \to E^1_{d+1,s}$ also leads to a map $H_s(\mathrm{Stab}(\sigma_{d+1}), M) \to H_s(\mathrm{Stab}(\sigma_d), M)$ given by the formula $\sum_{i=1}^{d+2} (-1)^{i+1} H_s(\mathrm{Int}(m_{i,d+1}), l(m_{i,d+1}))$.

4.3. Now let M be an abelian group A on which G acts trivially. Then $H_0(\mathrm{Stab}_G(\sigma_k), A) = A$ and $H_0(\mathrm{Int}(m_{ik}))$ is the identity. Thus we find:

Proposition. (i) $E^2_{r0} = 0$ *for* $0 \leqq r \leqq d+1$.

(ii) *For* $0 \leqq r \leqq d+\varepsilon$, $s \geqq 0$, *the differential* $E^1_{r+1,s} \to E^1_{r,s}$ *equals* $\sum_{i=1}^{r+1} (-1)^{i+1} \cdot H_s(\mathrm{Int}(m_{ir}))$.

4.4. Say $G = G_n^{PQ}$ is as in Example 4 of 3.5. We have $\sigma_r = (e_t, e_t + e_{t+1}, \dots, e_t + e_{t+r})$ and a reasonable choice for m_{ir} seems to be as follows. When $i > 1$ define $g \in GL_n(R)$ by $g(e_k) = e_k$ for $k \in [1, t+i-2] \cup [t+r+1, n]$, $g(e_k) = e_{k+1}$ for $t+i-1 \leqq k \leqq t+r-1$, $g(e_{t+r}) = (-1)^{r-i+1} e_{t+i-1}$. Then $g \in G_n^{PQ}$ (compute the determinant) and in terms of the left action we have $g\sigma_{r-1} = \partial_i(\sigma_r)$. In terms of the right action this means $\sigma_{r-1} g^{-1} = \partial_i(\sigma_r)$, so let us put $m_{ir} = g^{-1}$. When $i=1$ define $g \in GL_n(R)$ by $g(e_k) = e_k$ for $k \in [1, t-1] \cup [t+r+1, n]$, $g(e_t) = e_t + e_{t+1}$, $g(e_k) = e_{k+1} - e_{t+1}$ for $t+1 \leqq k \leqq t+r-1$, $g(e_{t+r}) = (-1)^{r+1} e_{t+1}$. As g is now an element of G_n^{PQ} with $g\sigma_{r-1} = \partial_1(\sigma_r)$, we put $m_{1r} = g^{-1}$. Observe that with these choices the m_{ir} centralize $Sq(\mathrm{Stab}_G(\sigma_r)) = G_n^{[1,t+r][1,t+r]}$. (Notation as in 3.2.) Recall that all inclusion maps are denoted inc and consider the composite of $H_s(\mathrm{inc})\colon H_s(Sq(\mathrm{Stab}_G(\sigma_r)), A) \to E^1_{r+1,s}$ with the differential $E^1_{r+1,s} \to E^1_{r,s}$ for some r, s with $s \geqq 0$, $0 \leqq r \leqq d+\varepsilon$. By Proposition 4.3 this composite equals $\sum_{i=1}^{r+1} (-1)^{i+1} H_s(\mathrm{Int}(m_{ir}))$, and, as the $\mathrm{Int}(m_{ir})$ coïncide with inc on $Sq(\mathrm{Stab}_G(\sigma_r))$, all terms in the sum cancel when r is odd while one term $H_s(\mathrm{inc})$ remains when r is even. When moreover $H_s(\mathrm{inc})\colon H_s(Sq(\mathrm{Stab}_G(\sigma_r)), A) \to E^1_{r+1,s}$ is surjective, this implies that the differential $E^1_{r+1,s} \to E^1_{r,s}$ vanishes for r odd and equals $H_s(\mathrm{inc})\colon H_s(\mathrm{Stab}_G(\sigma_r), A) \to H_s(\mathrm{Stab}_G(\sigma_{r-1}), A)$ for r even. We have derived this crucial fact only for a special choice of $G = G_n^{PQ}$, but it holds more generally:

4.5. **Lemma.** *In the situation of* 4.3 *let* r, s *be such that* $H_s(\mathrm{inc})\colon H_s(Sq(\mathrm{Stab}_G(\sigma_r)), A) \to H_s(\mathrm{Stab}_G(\sigma_r), A)$ *is surjective*, $s \geqq 0$, $1 \leqq r \leqq d+\varepsilon$. *When* r

is odd the differential $E^1_{r+1,s} \to E^1_{r,s}$ *vanishes and when r is even it equals*

$$H_s(\text{inc})\colon H_s(\text{Stab}_G(\sigma_r), A) \to H_s(\text{Stab}_G(\sigma_{r-1}), A).$$

Proof. We may assume that G_n^{PQ} is as in one of the examples in 3.5. In case of the fourth example, see 4.4. What we need to show in the other cases is that we may choose the m_{ir} within the centralizer of $Sq(\text{Stab}_G(\sigma_r))$, just as in 4.4. For the third example, compare with the fourth by taking transposes. The second example is subsumed by the fourth and the first one is not difficult.

Remarks. 1. Note that we have used our explicit choice of the σ_r, among other things to see that $\text{Stab}_G(\sigma_r)$ is contained in $\text{Stab}_G(\sigma_{r-1})$.

2. The map inc: $Sq(K) \to K$ is a split injection, for all $K \in \mathscr{F}_n$, in particular for $K = \text{Stab}_G(\sigma_r)$. (Check the examples in 3.5.) So $H_s(Sq(K), A) \to H_s(K, A)$ is always injective.

4.6. *Proof of Stability.* Lemma 4.5 and Proposition 3.7 together force stability. For, let m be such that for each P, Q as in 3.2 and each s with $0 \leqq s \leqq m$ the map $H_s(\text{inc})\colon H_s(G_n^{PQ}, A) \to H_s(XL(R), A)$ is an isomorphism for n sufficiently large. (Clearly $m=0$ is an example.) Fix some P, Q and consider the spectral sequence $E(P, Q; n)$ with $M = A$, cf. 3.7, 4.3. For $1 \leqq s \leqq m$, $0 \leqq r \leqq m+2$, r even and n sufficiently large the maps $H_s(\text{inc})\colon H_s(Sq(\text{Stab}_G(\sigma_r)), A) \to H_s(\text{Stab}_G(\sigma_r), A)$ and $H_s(\text{inc})\colon H_s(\text{Stab}_G(\sigma_r), A) \to H_s(\text{Stab}_G(\sigma_{r-1}), A)$ are isomorphisms. Also, d grows with n, so that by 4.5 the E^2 term of $E(P, Q; n)$ looks like Fig. 1 when n is sufficiently large.

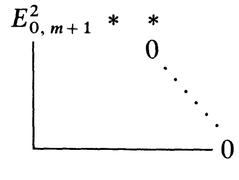

Fig. 1

We see from Fig. 1 that $E^2_{0,m+1} \cong E^r_{0,m+1}$ for $r \geqq 2$. As $E(P, Q; n)$ converges to zero, it follows that $E^2_{0,m+1} = 0$. But $E^2_{0,m+1}$ is the cokernel of $H_{m+1}(\text{inc})\colon H_{m+1}(\text{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$, so this $H_{m+1}(\text{inc})$ is surjective. Varying P, Q we get a lot of surjective maps this way and we see (cf. 3.3) that $H_{m+1}(Sq(G_n^{PQ}), A) \to H_{m+1}(G_n^{PQ}, A)$ is surjective for n sufficiently large, with the bound depending on P, Q, m. Now fix P, Q again and look back at the spectral sequence $E(P, Q; n)$. For n sufficiently large the E^2 term looks like Fig. 2, with $E^2_{1,m+1}$ equal to the kernel of the surjective map $H_{m+1}(\text{inc})\colon H_{m+1}(\text{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$, by Lemma 4.5.

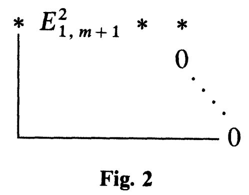

Fig. 2

We see from Fig. 2 that $E^2_{1,m+1} \simeq E^r_{1,m+1}$ for $r \geqq 2$. As $E(P, Q; n)$ converges to zero, $E^2_{1,m+1}$ must vanish and the map $H_{m+1}(\text{inc})$ must also be injective. Varying P, Q again and noting that a stabilization map $I: G_n^{\Phi\Phi} \to G_{n+1}^{\Phi\Phi}$ only differs by a change of co-ordinates from a map inc: $G_{n+1}^{\{1\}\{1\}} \to G_{n+1}^{\Phi\Phi}$, we see that for each P, Q as in 3.2 and n sufficiently large the map $H_{m+1}(\text{inc})$: $H_{m+1}(G_n^{PQ}, A) \to H_{m+1}(XL(R), A) = \lim\limits_{\vec{s}} H_{m+1}(G_s^{\Phi\Phi}, A)$ is an isomorphism. Thus stability holds by induction on m.

4.7. Clearly the argument in 4.6 also gives a method for obtaining quantitative results. One may try to improve a little on the ensuing bounds by means of an ad hoc analysis of the stability problem for homology in very low degrees. How much of an improvement one obtains depends on R, A and the effort made. The proof of the following theorem provides an example of such an approach. For other examples see [18], Chap. IV.

4.8. Conventions are as in 2.2, 3.2, 4.3.

Theorem. (Stability with constant coefficients).
Put $e = \max(1, \text{sdim})$. *The maps*

$$H_m(\text{inc}): H_m(Sq(G_n^{PQ}), A) \to H_m(G_n^{PQ}, A)$$

and

$$H_m(\text{inc}): H_m(G_n^{PQ}, A) \to H_m(XL(R), A)$$

are surjective when the size of $Sq(G_n^{PQ})$ *is at least* $2m+e-1$, *bijective when it is at least* $2m+e$ $(m \geqq 0)$.

Proof. We will go through the argument of 4.6, making some assumptions in order to guarantee that the induction step remains valid in the quantitative form corresponding with the theorem. Thereafter we will take care of the situations where one of the assumptions fails. So let m be a non-negative integer such that for $0 \leqq s \leqq m$ the maps $H_s(\text{inc}): H_s(Sq(G_n^{PQ}), A) \to H_s(G_n^{PQ}, A)$ and $H_s(\text{inc}): H_s(G_n^{PQ}, A) \to H_s(XL(R), A)$ are surjective when the size of $Sq(G_n^{PQ})$ is at least $2s+e-1$, bijective when it is at least $2s+e$. Our first task is to show that $H_{m+1}(\text{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$ is surjective when $Sq(\text{Stab}_G(\sigma_0))$ has size at least $2m+e+1$ (notations as in 3.7). Assumption 1 is that d is at least $m+1$. It makes that Proposition 4.3 applies where we need it: We see that $E^2_{m+2,0} = 0$. As $Sq(\text{Stab}_G(\sigma_0))$ is assumed to have size at least $2m+e+1$, one checks that the $Sq(\text{Stab}_G(\sigma_r))$ have size at least $2m+e+1-r$ for $0 \leqq r \leqq d+1$. By Lemma 4.5 this implies that the differentials $E^1_{r+1,s} \to E^1_{r,s}$ are given by $H_s(\text{inc})$ when r is even, $r+s \leqq m+2$, $r \geqq 2$, $1 \leqq s \leqq m$. (Use that assumption 1 implies $r \leqq d+\varepsilon$, and note that $2m+e+1-r \geqq 2s+e-1$). To see that the spectral sequence looks like Fig. 1 at the E^2 level, we observe that the differentials $E^1_{r+1,s} \to E^1_{r,s}$ are surjective when r is even, $2 \leqq r \leqq m+1$, $r+s = m+2$, injective when r is even, $2 \leqq r \leqq m$, $r+s = m+1$. (Note that $E^1_{rs} \cong H_s(XL(R), A)$ because $2m+e+2-r \geqq 2s+e$). We find (cf. 4.6) that $E^2_{0,m+1}$ vanishes, so that $H_{m+1}(\text{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$ is surjective. Forget for a moment that we needed to make assumption 1. Varying P, Q, n we get many surjective maps and find that indeed $H_{m+1}(Sq(G_n^{PQ}), A) \to H_{m+1}(G_n^{PQ}, A)$ is surjective as soon as $Sq(G_n^{PQ})$ has size at least $2m+e+1$ (cf. 3.3).

We also see that $H_{m+1}(Sq(G_n^{PQ}), A) \to H_{m+1}(XL(R), A)$ is surjective and hence that $H_{m+1}(G_n^{PQ}, A) \to H_{m+1}(XL(R), A)$ is surjective, when $Sq(G_n^{PQ})$ has such size. Next consider the spectral sequence $E(P, Q; n)$ when $Sq(\mathrm{Stab}_G(\sigma_0))$ has size at least $2m+e+2$. Now we make assumption 2, which says that d is at least $m+2$. We find that $E^2_{m+3,0}=0$. The $Sq(\mathrm{Stab}_G(\sigma_r))$ have size at least $2m+e+2-r$ for $0 \leqq r \leqq d+1$. The differential $E^1_{r+1,s} \to E^1_{r,s}$ equals $H_s(\mathrm{inc})$ when r is even, $r+s \leqq m+3$, $r \geqq 2$, $1 \leqq s \leqq m$; and it vanishes when $r=1$, $s=m+1$. (Use that $r \leqq d+\varepsilon$, $2m+e+2-r \geqq 2s+e-1$.) To see that the spectral sequence looks like Fig. 2 at the E^2 level, we note that the differentials $E^1_{r+1,s} \to E^1_{r,s}$ are surjective when r is even, $r \geqq 2$, $r+s=m+3$, $1 \leqq s \leqq m$; injective when r is even, $r \geqq 2$, $r+s=m+2$, $1 \leqq s \leqq m$. (Note that E^1_{rs} is isomorphic with $H_s(XL(R), A)$ because $2m+e+3-r \geqq 2s+e$.) We find that $E^2_{1,m+1}$ vanishes (cf. 4.6) and that $H_{m+1}(\mathrm{inc})$: $H_{m+1}(\mathrm{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$ is injective. As we know already that this $H_{m+1}(\mathrm{inc})$ is surjective it is an isomorphism. Forgetting that we used assumption 2, and varying P, Q, n again we can finish the induction step. Remains to consider the situations where assumption 1 or 2 fails.

4.9. Say assumption 1 fails, so that $d \leqq m$. In the context in which we made this assumption the size of $Sq(\mathrm{Stab}_G(\sigma_0))$ is at least $2m+e+1$. The size of $Sq(G)$ is $n-t$ and, when G is not square the size of $Sq(\mathrm{Stab}_G(\sigma_0))$ equals the size of $Sq(G)$. (Recall that $\mathrm{Stab}_G(\sigma_0)$ is a good predecessor of G.) When G is square, the size of $Sq(\mathrm{Stab}_G(\sigma_0))$ is one less than the size of $Sq(G)$. So in either case $Sq(\mathrm{Stab}_G(\sigma_0))$ has size $n-t-1+\varepsilon$, which should be at least $2m+e+1$. We get $m \geqq d = n-t-2-\mathrm{sdim} \geqq 2m+e-\varepsilon-\mathrm{sdim}$, hence $m \leqq \mathrm{sdim}-e+\varepsilon \leqq \varepsilon \leqq 1$, so m is 1 or 0 (m is non-negative integer). First say $m=1$. Then we must have $\varepsilon=1$, $d=1$. Looking at the spectral sequence $E(P, Q; n)$ we find that we are saved because m is odd and $\varepsilon=1$: The differential $E^1_{30} \to E^1_{20}$ is an isomorphism and the differential $E^1_{31} \to E^1_{21}$ is surjective, so the E^2 term is still described by Fig. 1. Next say $m=0$. Then we have $n-t \geqq 2+e-\varepsilon$, so when $\varepsilon=1$ one sees that $n-t \geqq 2$ and when $\varepsilon=0$ one sees that $n-t \geqq \max(2+\mathrm{sdim}, 3)$. We have to show that $H_1(\mathrm{Stab}_G(\sigma_0), A) \to H_1(G, A)$ is surjective. For any group K one has $H_1(K, A) = (K/[K, K]) \otimes_{\mathbb{Z}} A$, so we may assume $A=\mathbb{Z}$. When $n-t \geqq 2$, an easy exercise, well-known in classical algebraic K-theory, shows that $H_1(Sq(G)) \to H_1(G)$ is surjective. As $Sq(\mathrm{Stab}_G(\sigma_0))$ contains $Sq(G)$ when $\varepsilon=1$, we may now turn to the case $\varepsilon=0$, $n-t \geqq \max(2+\mathrm{sdim}, 3)$. Surjective stability for K_1 ([3], Thm. 4.2b, cf. 1.3) implies that G is generated by $\mathrm{Stab}_G(\sigma_0)$ and $E(R) \cap G$. But $E(R) \cap G$ is its own commutator subgroup, as $n-t \geqq 3$. Done.

4.10. Remains the case that assumption 2 fails, so that $d \leqq m+1$. Now the context tells that $Sq(\mathrm{Stab}_G(\sigma_0))$ has size at least $2m+e+2$ and we have to show that $H_{m+1}(\mathrm{Stab}_G(\sigma_0), A) \to H_{m+1}(G, A)$ is injective. As in 4.9 we find $m+1 \geqq d = n-t-2-\mathrm{sdim} \geqq 2m-\varepsilon+1$, and $m \leqq \varepsilon \leqq 1$, so $m=0$ or $m=1$. If $m=0$ use stability for K_1 again ([29], cf. 1.3). If $m=1$ then $\varepsilon=1$, $d=2$ and one checks that the spectral sequence $E(P, Q; n)$ has all the properties we need. (To see that E^2_{40} vanishes, use that the boundary map $C_4 \to C_3$ factors over $Z_3 = L_5$ and compare with 4.2). End of proof of Theorem 4.8.

4.11. As a corollary to Theorem 4.8 we have

Theorem. *Let* $e = \max(1, \mathrm{sdim})$. *Then* $H_m(GL_n(R)) \to H_m(GL_{m+1}(R))$ *and* $H_m(E_n(R)) \to H_m(E_{n+1}(R))$ *are surjective for* $n \geqq 2m+e-1$, *injective for* $n \geqq 2m+e$ $(m \geqq 0)$.

Proof. We have put $A = \mathbb{Z}$ and $P = Q = \varnothing$ in Theorem 4.8. Taking $XL(R) = GL(R)$ the statements for the $H_m(GL_i(R))$ follow directly from 4.8. Next take $XL(R) = E(R)$. By injective stability for K_1 ([29], *cf.* 1.3) the group $G_n^{\varnothing\varnothing}$ equals $E_n(R)$ for $n \geqq \mathrm{sdim}+2$. Therefore the statements follow from 4.8 for $m \geqq 2$. But for $m=1$ the behaviour of $H_m(E_n(R))$ is rather simple: It vanishes for $n \geqq 3$. Similarly the case $m=0$ poses no problem.

4.12. For $n \geqq 3$ the plus construction may be applied to $BGL_n(R)$ with respect to the normal closure of $E_n(R)$. (Under mild conditions $E_n(R)$ is its own normal closure. For instance, when $n \geqq e+2$ then $E_n(R)$ is the commutator subgroup of $GL_n(R)$. See [29].)

Corollary. $\pi_m(BGL_n^+(R)) \to \pi_m(BGL_{n+1}^+(R))$ *is surjective for* $n \geqq \max(3, 2m+e-1)$, *injective for* $n \geqq 2m+e+1$ $(m \geqq 1)$.

Proof. This proof is well-known. For $m=1$ the result itself is classical (cf. 1.3). For $m \geqq 2$, recall that $BE_n^+(R)$ is the universal covering space of $BGL_n^+(R)$, at least for $n \geqq 2m+e-1$. (See [17], Prop. 1.1.7.) As $H_m(BE_n^+(R)) = H_m(E_n(R))$, the result follows from Theorem 4.11 by a Hurewicz argument.

4.13. *Remarks.* 1. Note that Corollary 4.12 is weaker than Theorem 4.11 in that it does not give injectivity for $n = 2m+e$. The reason is that the Hurewicz argument requires the vanishing of relative homology groups. The relative homology group corresponding with injectivity for the case $n = 2m+e$ would be $H_{m+1}(E_{2m+e+1}(R), E_{2m+e}(R))$, but in general $H_{m+1}(E_{2m+e}(R)) \to H_{m+1}(E_{2m+e+1}(R))$ is not surjective. (Example: $R = \mathbb{Z}$, $e = m = 1$). Thus the vanishing of relative homology groups does not tell everything about the range of stability for the absolute homology groups.

2. Using Vaserstein's theory of "big" modules, our proof can be generalized to yield stability for $H_m(GL(M \oplus R^n)) \to H_m(GL(M \oplus R^{n+1}))$, where M is a fixed right R-module. (Compare 2.7, Remark 2.) One will get an isomorphism for n beyond a certain bound which depends on m and sdim. One may also use a better bound which depends on m, R and M. It is an easy consequence of such a result that, when P is a finitely generated projective R-module, the maps $H_m(GL(M \oplus R^n)) \to H_m(GL(M \oplus R^n \oplus P))$ are isomorphisms for n beyond a certain bound which depends on m and sdim (or a sharper bound depending on m, R, M and $M \oplus P$). In particular one can thus obtain something like the main result in Charney [7].

§5. Stability for Twisted Coefficients

5.1. In this section we prove stability for homology with twisted coefficients, when these twisted coefficients form a strongly central coefficient system of finite degree. (This terminology is explained below.)

5.2. Let us write G_n for the square group $G_n^{\Phi\Phi}=GL_n(R)\cap XL(R)$ (cf. 3.2). We recall some notions from Dwyer [9]. A *coefficient system* ρ for $\{G_n\}_{n\geqq 0}$ is a sequence of left G_n-modules ρ_n together with I-linear maps F_n: $\rho_n\to\rho_{n+1}$, where I: $G_n\to G_{n+1}$ denotes (upper) inclusion, as usual (cf. 3.2, 3.8). Fix $\gamma\in\{-1,1\}$ such that $\begin{pmatrix}0 & \gamma\\ 1 & 0\end{pmatrix}\in G_2$. (When $XL(R)=GL(R)$ one may take $\gamma=1$. This is Dwyer's choice.) For $n\geqq 2$ define $s_n\in G_n$ by $s_n(e_i)=e_i$ for $1\leqq i\leqq n-2$, $s_n(e_{n-1})=e_n$, $s_n(e_n)=\gamma e_{n-1}$. A coefficient system is called *central* when for each n the image of ρ_n under $F_{n+1}F_n$ is pointwise fixed by $l(s_{n+2})$: $\rho_{n+2}\to\rho_{n+2}$. (Recall that, when M is a G-module and $g\in G$, the map $m\mapsto gm$ from M to M is denoted $l(g)$.) We call a coefficient system *strongly central* when for each n the image of ρ_n under $F_{n+1}F_n$ is pointwise fixed by both $l(s_{n+2})$ and $l(e_{n+1,n+2})$, where $e_{n+1,n+2}$ is the elementary matrix which sends e_{n+2} to $e_{n+2}+e_{n+1}$ and fixes all other e_i.

5.3. Put $c_n=s_2s_3\dots s_n$. Then $c_n(e_i)=e_{i+1}$ for $1\leqq i\leqq n-1$, $c_n(e_n)=\pm e_1$. The map $J=\mathrm{Int}(c_{n+1})$: $G_n\to G_{n+1}$ is called the *lower inclusion* map. ($\mathrm{Int}(c_{n+1})(g)=c_{n+1}gc_{n+1}^{-1}$, cf. 4.2.) The lower inclusion map shifts a matrix one place down the diagonal. (Check this.) Let ρ be a central coefficient system. We denote by $J^*(\rho_{n+1})$ the representation of G_n which has the same underlying abelian group as ρ_{n+1} and a G_n-action obtained from the G_{n+1}-action by "restriction" via J. One defines a *shifted* coefficient system $\Sigma\rho$ by $(\Sigma\rho)_n=J^*(\rho_{n+1})$, with the structure map $(\Sigma\rho)_n\to(\Sigma\rho)_{n+1}$ equal to the structure map F_{n+1} of ρ. We will use notations such as F_n, $l(s_n)$ in their original sense, not in a shifted sense. When μ: $\rho\to\rho'$ is a coefficient system map, then $(\Sigma\mu)_n=\mu_{n+1}$ defines a coefficient system map $\Sigma\mu$ from $\Sigma\rho$ to $\Sigma\rho'$.

5.4. The map $\tau_n=l(c_{n+1})F_n$: $\rho_n\to J^*(\rho_{n+1})$ is G_n-linear. (It is the composite of G_n-linear maps F_n: $\rho_n\to I^*(\rho_{n+1})$ and $l(c_{n+1})$: $I^*(\rho_{n+1})\to J^*(\rho_{n+1})$.) We have $\tau_{n+1}F_n=l(c_{n+1})\,l(s_{n+2})\,F_{n+1}F_n$, which equals $l(c_{n+1})\,F_{n+1}F_n$ because ρ is central. So $\tau_{n+1}F_n=F_{n+1}\,l(c_{n+1})\,F_n=F_{n+1}\tau_n$, hence:

Lemma (cf. [9]). *When ρ is a central coefficient system then τ_n: $\rho_n\to(\Sigma\rho)_n$ are the constituents of a coefficient system map τ: $\rho\to\Sigma\rho$. Hence kernel(τ) and cokernel(τ) are coefficient systems.*

5.5. Often $\Sigma\rho$ is the direct sum of two subsystems, one of which is isomorphic with ρ via τ. (The other one is then isomorphic with cokernel(τ), of course.) When this is the case we say that *$\Sigma\rho$ splits* and that cokernel(τ) is the *system of differences* of ρ, notation $\Delta\rho$. Now let us explain what we mean by saying that a strongly central coefficient system ρ has *degree k*. The definition is inductive. When $k<0$ we mean that ρ is the zero system. When $k\geqq 0$ we mean that $\Sigma\rho$ splits and that $\Delta\rho$ is a strongly central coefficient system of degree $k-1$. Examples can be obtained as in ([9], Sect. 3) from a functor of finite degree. For instance, when R is commutative one may take $\rho_n=\overset{k}{\bigotimes}(R^n)$, the k-th tensor power over R of the standard representation of G_n in R^n. This ρ has degree k. (Check this, and pay attention to the structure maps in order to appreciate the definition of τ.)

5.6. We are interested in the relative homology groups $\mathrm{Rel}^n_m(\rho)=H_m(G_{n+1}, G_n; \rho_{n+1}, \rho_n)$, based on I and F_n (cf. 3.8).

Theorem. *Let ρ be a strongly central coefficient system of degree k. Then $H_m(G_{n+1}, G_n; \rho_{n+1}, \rho_n)$ vanishes for $n \geqq 2m+\mathrm{sdim}+k$. $(m \geqq 0)$.*

Remark. One can often improve a little on the range given in the theorem. In this respect the situation is similar to the one discussed in 4.7. In the other direction, one may smoothen the proof when $2m+\mathrm{sdim}+k$ is replaced by $2m+\mathrm{sdim}+k+2$. Then the ad hoc arguments for homology of degree zero may be deleted. (In particular, the references to remark 3.15 become unnecessary.)

5.7. Dwyer uses a qualitative approach. He calls a system ρ *stable* when for each m the $\mathrm{Rel}^n_m(\rho)$ (cf. 5.6) vanish for n sufficiently large. Further he calls ρ *strongly stable* when for each $j \geqq 0$ the system $\Sigma^j \rho$ is stable. In those terms we have (compare [9], Thm. 2.2):

Theorem. *Let ρ be a central coefficient system, $\tau\colon \rho \to \Sigma\rho$ as in 5.4. When kernel(τ) and cokernel(τ) are strongly stable, so is ρ.*

5.8. First let us prove Theorem 5.7. (The proof will have a lot in common with [9].) Let m be a non-negative integer so that for $0 \leqq s < m$ and all P, Q as in 3.2 the $H_s(G^{PQ}_{n+1}, G^{PQ}_n; \rho_{n+1}, \rho_n)$ vanish for n sufficiently large. Here ρ_n is viewed as a G^{PQ}_n-module via inc: $G^{PQ}_n \to G_n$, of course. And one uses $I\colon G^{PQ}_n \to G^{PQ}_{n+1}$ together with $F_n\colon \rho_n \to \rho_{n+1}$ to define the relative homology groups. We will argue by induction on m.

Claim. The map $g^n_m\colon \mathrm{Rel}^n_m(\rho) \to \mathrm{Rel}^{n+1}_m(\rho)$, induced by the lower inclusion maps and τ, is surjective for n sufficiently large.

Remark. The map $\mathrm{Rel}^n_m(\rho) \to \mathrm{Rel}^{n+1}_m(\rho)$ which is induced by upper inclusions and the structure maps F_i is always zero because it factors over $H_m(G_{n+1}, G_{n+1}; \rho_{n+1}, \rho_{n+1})=0$. This explains why g^n_m is defined the way it is. We will exploit the similarity between g^n_m and the zero map in 5.10 (cf. [9]).

5.9. To prove the claim we will break up g^n_m into a chain of maps, and prove surjectivity for each of them. For n sufficiently large $H_{m-1}(G_{n+1}, G_n; \mathrm{kernel}(\tau_{n+1}), \mathrm{kernel}(\tau_n))$ and $H_m(G_{n+1}, G_n; \mathrm{cokernel}(\tau_{n+1}), \mathrm{cokernel}(\tau_n))$ vanish. As in ([9], proof of lemma 2.8) we may plug this into two long exact sequences for relative homology (see Lemma 3.11) to find that τ induces a surjective map $\mathrm{Rel}^n_m(\rho) \to \mathrm{Rel}^n_m(\Sigma\rho)$. Clearly the lower inclusion maps induce an isomorphism from $\mathrm{Rel}^n_m(\Sigma\rho)$ to $H_m(JG_{n+1}, JG_n; \rho_{n+2}, \rho_{n+1})=H_m(G^{\{1\}\{1\}}_{n+2}, G^{\{1\}\{1\}}_{n+1}; \rho_{n+2}, \rho_{n+1})$. Now take P, Q as in 3.2, $M=\rho_n$, $M'=\rho_{n+1}$ and consider the spectral sequence $E(P, Q; n+1, n)$ of 3.14. (So $G=G^{PQ}_n$, $G'=G^{PQ}_{n+1}$ etc.) For n sufficiently large its E^1 term looks like Fig. 3. (Note that d grows with n.)

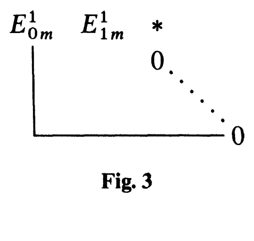

Fig. 3

We see from Fig. 3 that $E^r_{0m} \simeq \text{cokernel}\,(E^1_{1m} \to E^1_{0m})$ for $r \geqq 2$. As the spectral sequence converges to zero, the differential $E^1_{1m} \to E^1_{0m}$ must be surjective. By Proposition 3.14 this differential is the natural map from $H_m(\text{Stab}_{G'}(\sigma_0), \text{Stab}_G(\sigma_0); \rho_{n+1}, \rho_n)$ to $H_m(G', G; \rho_{n+1}, \rho_n)$.

Varying P, Q we get a lot of surjective maps this way and we see in particular (cf. 3.3) that $H_m(G^{\{1\}\{1\}}_{n+2}, G^{\{1\}\{1\}}_{n+1}; \rho_{n+2}, \rho_{n+1}) \to \text{Rel}^{n+1}_m(\rho)$ is surjective for n sufficiently large. All in all g^n_m is indeed surjective for n sufficiently large. The importance of this fact is explained by:

5.10. **Lemma.** *When g^n_m and g^{n-1}_m are surjective then* $\text{Rel}^{n+1}_m(\rho)$ *vanishes.*

Proof. The surjective mapping g^n_m fits into a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots\; H_m(G_{n+1}, \rho_{n+1}) & \xrightarrow{a_1} & \text{Rel}^n_m(\rho) & \xrightarrow{a_2} & H_{m-1}(G_n, \rho_n) & \xrightarrow{l'_2} \cdots \\
\downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle g^n_m} & & \downarrow{\scriptstyle l_2} & \\
\xrightarrow{l'_1} H_m(G_{n+2}, \rho_{n+2}) & \xrightarrow{b_1} & \text{Rel}^{n+1}_m(\rho) & \xrightarrow{b_2} & H_{m-1}(G_{n+1}, \rho_{n+1}). &
\end{array}$$

The map l_1 is induced by J and τ, the map l'_1 by I and F_{n+1}. So l_1 is obtained from l'_1 by composition with the map $H_m(\text{Int}(c_{n+2}), l(c_{n+2}))$ from $H_m(G_{n+2}, \rho_{n+2})$ to itself. But it is well known that $H_m(\text{Int}(x), l(x))$ always equals the identity on $H_m(K, M)$, when $x \in K$. ("Inner conjugation acts trivially on homology".) So l_1 equals l'_1 and, similarly, l_2 equals l'_2. It follows that $b_2 g^n_m = l_2 a_2 = l'_2 a_2 = 0$, and, as g^n_m is surjective, b_2 equals 0. We find that b_1 is surjective. Similarly a_1 is surjective. But $g^n_m a_1 = b_1 l_1 = b_1 l'_1 = 0$, so we get a map from $H_m(G_{n+1}, \rho_{n+1})$ to $\text{Rel}^{n+1}_m(\rho)$ which is both surjective and zero.

Remark. When $m=0$ the map a_1 is always surjective so that surjectivity of g^n_0 already implies the vanishing of $\text{Rel}^{n+1}_0(\rho)$.

5.11. We now know that $\text{Rel}^n_m(\rho)$ vanishes for n sufficiently large. For $t \geqq 0$ and n sufficiently large $\Sigma^t \tau: \Sigma^t \rho \to \Sigma^{t+1} \rho$ induces a surjection $\text{Rel}^n_m(\Sigma^t \rho) \to \text{Rel}^n_m(\Sigma^{t+1} \rho)$, for the same reasons as in 5.9, where we discussed the special case $t=0$.

So the $\text{Rel}^n_m(\Sigma^t \rho)$ also vanish, for n sufficiently large with respect to m and t. This amounts to the same as the vanishing of the $H_m(G^{TT}_{n+t+1}, G^{TT}_{n+t}; \rho_{n+t+1}, \rho_{n+t})$ for large n, where $T=[1,t]$. Using the surjective maps obtained in 5.9 one concludes (cf. 3.3) that for all P, Q the $H_m(G^{PQ}_{n+1}, G^{PQ}_n; \rho_{n+1}, \rho_n)$ vanish eventually. Thus Theorem 5.7 holds by induction on m.

5.12. To prove Theorem 5.6 we first take a closer look at the differential $E^1_{2m} \to E^1_{1m}$ in the spectral sequence $E(P, Q; n+1, n)$ when (P, Q) equals $(\varnothing, \varnothing)$ or $(\{1\}, \varnothing)$. (When we would not take a closer look at this differential, but just would mimic the proof of Theorem 5.7, the result would be a bound like $3m + \text{sdim} + 2k$, rather than $2m + \text{sdim} + k$). As before we take $M = \rho_n$, $M' = \rho_{n+1}$ in 3.14. Inspecting the proof of Proposition 4.3 we see that the differential $E^1_{2m} \to E^1_{1m}$ is the difference of two maps, one being induced by $\text{Int}(m_{11})$ and $l(m_{11})$ (notations as in 4.2), the other by $\text{Int}(m_{21})$ and $l(m_{21})$. (Assume $d + \varepsilon \geqq 1$.) For a strongly central ρ it is easy to see that for any g in the subgroup generated by e_{12} and s_2 the map $l(g)$ fixes the image of ρ_{n-2} under

$(\Sigma\tau)_{n-2}\tau_{n-2}=\tau_{n-1}\tau_{n-2}=l(c_n)F_{n-1}l(c_{n-1})F_{n-2}=l(c_nc_{n-1})F_{n-1}F_{n-2}$. In particular this applies to $g=m_{i1}$ when the m_{i1} are chosen sensibly (cf. proof of Lemma 4.5). This suggests the following analogue of Lemma 4.5.

5.13. **Lemma.** *Let* (P,Q) *equal* $(\emptyset,\emptyset)$ *or* $(\{1\},\emptyset)$, $d+\varepsilon\geqq 1$. *Let the map* $\mathrm{Rel}_m^{n-2}(\rho)\to H_m(\mathrm{Stab}_{G'}(\sigma_1), \mathrm{Stab}_G(\sigma_1); \rho_{n+1}, \rho_n)$, *induced by* JJ *and* $(\Sigma\tau)\tau$, *be surjective. Then the differential* $E^1_{2m}\to E^1_{1m}$ *in the spectral sequence* $E(P,Q;n+1,n)$ *vanishes.*

Proof. The composite of the given surjective map with the differential is the difference of two maps, each of which is induced by the same morphism in RelRep from $(I,F_{n-2})\colon(G_{n-2},\rho_{n-2})\to(G_{n-1},\rho_{n-1})$ to $(I,F_n)\colon(*,\rho_n)\to(*,\rho_{n+1})$.

Remark. Of course there are similar statements for other differentials $E^1_{r+1,s}\to E^1_{r,s}$, but we don't need them.

5.14. Just as in 5.8 we now replace Theorem 5.6 by a stronger statement, which will then be proved by induction on m and k. Namely, we claim that for P, Q as in 3.2 and $s\geqq 0$ the $H_s(G^{PQ}_{n+1}, G^{PQ}_n; \rho_{n+1}, \rho_n)$ vanish when $Sq(G^{PQ}_n)$ has size at least $2s+\mathrm{sdim}+k$ (ρ as in 5.6). Choose non-negative integers m and k so that this claim holds for systems of degree at most $k-1$ and also for ρ itself when $s\leqq m-1$. The group $\mathrm{Rel}^n_s(\Sigma^t(\varDelta\rho))$ may be viewed as $H_s(G^{TT}_{n+t+1}, G^{TT}_{n+t}; (\varDelta\rho)_{n+t+1}, (\varDelta\rho)_{n+t})$ with $T=[1,t]$, so it vanishes for $n\geqq 2s+\mathrm{sdim}+k-1$, $(t\geqq 0)$. It follows that $\Sigma^t\tau\colon\Sigma^t\rho\to\Sigma^{t+1}\rho$ induces a surjection from $\mathrm{Rel}^n_s(\Sigma^t\rho)$ to $\mathrm{Rel}^n_s(\Sigma^{t+1}\rho)$ for $n\geqq 2s+\mathrm{sdim}+k-1$, $t\geqq 0$. We may rewrite the target group as $H_s(G^{TT}_{n+t+2}, G^{TT}_{n+t+1}; \rho_{n+t+2}, \rho_{n+t+1})$ where $T=[1,t+1]$. For P, Q as usual we consider the spectral sequence $E(P,Q;\ n+1,n)$ again. (M, M' as before, cf. 5.9.) When $Sq(\mathrm{Stab}_G(\sigma_0))$ has size at least $2m+\mathrm{sdim}+k-1$, we wish to show that the spectral sequence looks like Fig. 3 again so that the natural map from $H_m(\mathrm{Stab}_{G'}(\sigma_0), \mathrm{Stab}_G(\sigma_0); \rho_{n+1}, \rho_n)$ to $H_m(G', G; \rho_{n+1}, \rho_n)$ must be surjective. When $m=0$ it is clear that this map is surjective for any coefficient system, so let us look at the case $m>0$. When $m+1\leqq d+\varepsilon+1$ all relevant E^1_{rs} terms can be interpreted by means of Proposition 3.14. Further $Sq(\mathrm{Stab}_G(\sigma_i))$ has size at least $2m+\mathrm{sdim}+k-1-i$ so that our assumptions make that one gets Fig. 3 indeed. In fact, by remark 3.15 it suffices to have $m+1\leqq d+\varepsilon+2$. Now suppose $m\geqq d+\varepsilon+2$. The size of $Sq(\mathrm{Stab}_G(\sigma_0))$ is $n-t-1+\varepsilon$ where $P\cup Q=[1,t]$, cf. 4.9, so $n-t-1+\varepsilon\geqq 2m+\mathrm{sdim}+k-1$. We find $m\geqq d+\varepsilon+2=n-t-\mathrm{sdim}+\varepsilon\geqq 2m+k\geqq 2m$, in contradiction with the assumption $m>0$. Thus we have obtained many surjective maps again.

5.15. We also like to have some injective maps. In particular, we wish to show that there is an injective map from $\mathrm{Rel}^{n-1}_m(\rho)$ to $\mathrm{Rel}^n_m(\rho)$ for $n\geqq 2m+\mathrm{sdim}+k+1$. As the $\mathrm{Rel}^n_m(\rho)$ vanish for n sufficiently large (see Theorem 5.7 or its proof, in particular Lemma 5.10) this will imply the vanishing of $\mathrm{Rel}^n_m(\rho)$ for $n\geqq 2m+\mathrm{sdim}+k$. As in 5.11 we may then exploit the many surjective maps obtained in 5.14 to establish the claim in 5.14 for $s=m$ (cf. 3.3). Thus induction on m will apply and the theorem will follow by induction on k. Remains to get the injective maps.

5.16. When $m=0$ we may use the remark in 5.10 and the surjective maps from 5.14 to see that $\mathrm{Rel}^n_m(\rho)$ vanishes for $n\geqq 2m+\mathrm{sdim}+k$. So we may assume $m\geqq 1$.

As ρ is a direct summand of $\Sigma\rho$ there is an injective map from $\mathrm{Rel}_m^{n-1}(\rho)$ to $\mathrm{Rel}_m^{n-1}(\Sigma\rho)\cong \mathrm{H}_m(G_{n+1}^{TT}, G_n^{TT};\ \rho_{n+1}, \rho_n)$, where $T=\{1\}$. Remains to show that the natural maps $H_m(G_{n+1}^{TT}, G_n^{TT};\ \rho_{n+1}, \rho_n)\to H_m(G_{n+1}^{T\Phi}, G_n^{T\Phi};\ \rho_{n+1}, \rho_n)$ and $H_m(G_{n+1}^{T\Phi}, G_n^{T\Phi};\ \rho_{n+1}, \rho_n)\to \mathrm{Rel}_m^n(\rho)$ are injective for $n\geqq 2m+\mathrm{sdim}+k+1$. In other words we need to show that for such n the differential $E_{1m}^1\to E_{0m}^1$ is injective in the spectral sequence $E(P, Q; n+1, n)$ where P, Q are as in Lemma 5.13. We have $d+\varepsilon=n-t-2-\mathrm{sdim}+\varepsilon=n-2-\mathrm{sdim}\geqq 2m+k-1\geqq m\geqq 1$ and it is easy to see from 5.14 that Lemma 5.13 applies. So E_{1m}^2 is isomorphic with the kernel of the differential which we wish to be injective. Note also that $m+2\leqq d+\varepsilon+2$. Using Proposition 3.14 and Remark 3.15 we derive from the inductive assumptions in 5.14 that the E^1 term looks like Fig. 4.

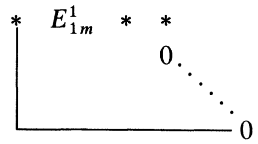

Fig. 4.

From Fig. 4 one sees that $E_{1m}^2\cong E_{1m}^r$ for $r\geqq 2$. As the spectral sequence converges to zero, E_{1m}^2 must vanish and we are through.

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