ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 258 (2002) 71–111

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Intersection cohomology of $B \times B$-orbit closures in group compactifications

T.A. Springer

Universiteit Utrecht, Dept. of Mathematics, PO Box 80010, NL-3508 Utrecht, The Netherlands

Received 18 July 2001

Dedicated to Claudio Procesi on the occasion of his 60th birthday

Introduction

An adjoint semi-simple group G has a "wonderful" compactification X, which is a smooth projective variety, containing G as an open subvariety. X is acted upon by $G \times G$ and, B denoting a Borel subgroup of G, the group $B \times B$ has finitely many orbits in X. The main results of this paper concern the intersection cohomology of the closures of the $B \times B$-orbits. Examples of such closures are the "large Schubert varieties," the closures in X of the double cosets BwB in G.

After recalling some basic results about the wonderful compactification, we discuss in Section 1 the description of the $B \times B$-orbits, and establish some basic results.

In Section 2 the "Bruhat order" of the set V of orbits is introduced and described explicitly. As an application we obtain cellular decompositions of the large Schubert varieties.

Let $\mathcal{H}$ be the Hecke algebra associated to G, it is a free module over an algebra of Laurent polynomials $\mathbb{Z}[u, u^{-1}]$. As a particular case of results of [MS], the spherical $G \times G$-variety X defines a representation of the Hecke algebra associated to $G \times G$, i.e. $\mathcal{H} \otimes_{\mathbb{Z}[u,u^{-1}]} \mathcal{H}$, in a free module $\mathcal{M}$ over an extension of $\mathbb{Z}[u, u^{-1}]$, with a basis (m_v) indexed by V. The definition of $\mathcal{M}$ is sheaf-theoretical, working over the algebraic closure of a finite field. This is discussed in Section 3. On the model of [LV] a duality map Δ is introduced on $\mathcal{M}$, coming from Verdier duality in sheaf theory. The matrix coefficients of Δ relative to the

E-mail address: springer@math.uu.nl.

0021-8693/02/$ – see front matter © 2002 Elsevier Science (USA). All rights reserved.
PII: S0021-8693(02)00515-X

basis (m_v) are discussed at the end of Section 3. They bear some resemblance to the R-polynomials of [KL].

In Section 4 it is shown that the intersection cohomology of an orbit closure $\bar{v}$ leads to "Kazhdan–Lusztig" elements in $\mathcal{M}$. The results about matrix coefficients of Section 3 together with results of [MS] imply the evenness of local intersection cohomology, and the existence of Kazhdan–Lusztig polynomials.

We also prove evenness of global intersection cohomology of closures $\bar{v}$. The results on intersection cohomology, proved in the first instance in positive characteristics, then also follow in characteristic 0, and over $\mathbb{C}$.

Section 5 contains a brief discussion of the extension of results of the previous sections to intersection cohomology of an orbit closure $\bar{v}$, for certain non-constant sheaves on v.

We have formulated the constructions of the paper (e.g. of the $\mathcal{H} \otimes \mathcal{H}$-module $\mathcal{M}$) in such a manner that they also make sense for general Coxeter groups. Section 6 contains some remarks about the constructions for such groups.

Computation by hand of our Kazhdan–Lusztig polynomials turns out to be quite cumbersome, the only manageable case (for the author) being $G = PGL_2$. The Appendix A by W. van der Kallen gives a number of numerical examples, obtained by computer calculations.

1. Preliminaries

In the sequel G is a connected, adjoint, semi-simple group over the algebraically closed field k. We denote by B and T a Borel group and a maximal torus contained in it. R is the root system of (G, T) and R^+ the system of positive roots of R defined by B. The Weyl group of R is W. For $w \in W$ we denote by $\dot{w}$ a representative in the normalizer $N(T)$.

We denote by S the set of simple reflections defined by R^+, and by D the set of simple roots. For $I \subset D$ let $W_I \subset W$ be the parabolic subgroup of W generated by the reflections in the roots of I. We write $S_I = S \cap W_I$.

Denote by $W^I = \{x \in W \mid x(I) \subset R^+\}$ the set of distinguished coset representatives of W/W_I and by $w_{0,I}$ the maximal element of W_I. On W and its subsets W_I and W^I we have the usual Bruhat orders.

1.1. We introduce the "wonderful" compactification X of G. We recall a number of results, established in [DS,B1]. (In [DS] it is assumed that $\mathrm{char}(k) \neq 2$. This restriction is necessary in the general situation discussed there, but is unnecessary for the compactification of G.)

X is an irreducible, smooth, projective $G \times G$-variety. It contains G as an open $G \times G$-stable subvariety (the action being $(g, h).x = gxh^{-1}$, for $g, h, x \in G$). The $G \times G$-orbits X_I in X are indexed by the subsets I of D.

Let P_I be the standard parabolic subgroup defined by $I \subset D$, the notation being such that the Levi subgroup L_I containing T has root system with basis I. We denote by C_I the center of L_I, by $G_I = L_I/C_I$ the corresponding adjoint group, by $B_I \subset G_I$ the image of $B \cap L_I$, and by T_I the image of T. Then B_I is a Borel group of G_I and T_I a maximal torus. Notice that C_I is connected (it is the intersection of the kernels of the simple roots in I and these form part of the basis D of the character group of T).

Let $B^- \supset T$ be the opposite of B and $P_I^- \supset B^-$ the opposite of P_I. Notice that L_I is a Levi subgroup of both P_I and P_I^-.

The G-orbit X_I is a $G \times G$-equivariant fiber space over $G/P_I^- \times G/P_I$, such that the fiber over $P_I^- \times P_I$ is G_I. In fact,

$$X_I = (G \times G) \times_{P_I^- \times P_I} G_I, \tag{1}$$

$P_I^- \times P_I$ acting on G_I via its quotient $G_I \times G_I$. Similarly,

$$\overline{X_I} = (G \times G) \times_{P_I^- \times P_I} \overline{G_I}.$$

We may view $\overline{G_I}$ to be the wonderful compactification of G_I.

X_I contains a unique base point h_I such that

(a) $(B \times B^-).h_I$ is dense in X_I,
(b) there is a cocharacter λ of T with $h_I = \lim_{t\to 0} \lambda(t)$ (see [B1, Proposition A1]).

Under the identification (1), h_I is the image in X_I of $(1, 1, 1)$. We have $h_D = 1$.

If H is an algebraic group denote by $R_u(H)$ its unipotent radical and by H_{diag} the diagonal of $H \times H$. It follows from (1) that the isotropy group of h_I in $G \times G$ is the semi-direct product of $R_u(P_I^-) \times R_u(P_I)$ and $(L_I)_{\mathrm{diag}}.(C_I \times \{1\})$ (cf. [B1, Proposition A1]).

If $I \subset D$ we define $I^* \subset D$ by $I^* = -w_{0,D}(I)$. Then $(I^*)^* = I$. If A is a subset of W we put $A^* = w_{0,D} A w_{0.D}$.

1.2. Lemma.
(i) *The isomorphism $g \mapsto g^{-1}$ of G extends to an isomorphism σ of the variety X such that for $g, h \in G$, $x \in X$*

$$\sigma.\big((g,h).x\big) = (h,g).(\sigma.x).$$

(ii) $\sigma(X_I) = X_{I^*}$.
(iii) $\sigma.h_I = (w_{0,D}, w_{0,D}).h_{I^*}$.

Proof. For (i) see [S, 1.2]; (ii) readily follows from the proof of (i); and (iii) is a consequence of the characterization of the points h_I given above. □

For $w \in W^I$ put $\iota_I(w) = w_{0,D} w w_{0,I}$. Then ι_I is a bijection of W^I and

$$l\big(\iota_I(w)\big) = l(w_{0,D}) - l(w_{0,I}) - l(w). \tag{2}$$

$B \times B$ has finitely many orbits in X. They are described in the following lemma.

1.3. Lemma.

(i) *The $B \times B$-orbits in X_I are of the form*

$$\mathcal{O} = (B \times B).(x, w).h_I, \tag{3}$$

with unique elements $w \in W$, $x \in W^I$.

(ii) $\dim \mathcal{O} = l(w_{0,D}) - l(x) + l(w) + |I|$.

(iii) $\sigma.\mathcal{O} = (B \times B).(\iota_{I^*}(y^*), \iota_{I^*}(x^*)))(w_{0,I^*}(z^*)^{-1} w_{0,I^*}).h_{I^*}$, *where $y \in W^I$, $z \in W_I$, and $w = yz$.*

(iv) *The isotropy group of $(x, w).h_I$ in $B \times B$ is the semi-direct product of a connected unipotent normal subgroup and the isotropy group in $T \times T$, which consists of the $(t, t') \in T \times T$ with $x^{-1}(t).w^{-1}(t')^{-1} \in C_I$.*

Proof. The $B \times B^-$-orbits in X_I are of the form

$$(B \times B^-).(v, y).h_I = (B \times B^-).(x, w).h_I,$$

where $x, y \in W^I$, $z \in W_I$, and $v = xz^{-1}$, $w = yz$ (by [B1, 2.1]). Hence the $B \times B$-orbits are of the form

$$(B \times B).(x, w_{0,D} yz).h_I.$$

This proves (i).

For $v = xz^{-1}$ we have

$$\mathcal{O}_1 = (B \times B^-).(v, y).h_I = (B \times B^-).(x, y).(B_I \times B_I^-).\big(1, z^{-1}\big).h_I \tag{4}$$

(see [B1, p. 151]). It follows that

$$\dim \mathcal{O}_1 = \dim BxP_I^- + \dim B^- y P_I + \dim B_I z^{-1} B_I^-.$$

Now $BxP_I^- = w_{0,D} B^- \iota_I(x) P_I^-$, whence $\dim BxP_I^- = l(\iota_I(x))$. Similarly, $\dim B^- y P_I = l(\iota_I(y))$. We conclude that

$$\dim \mathcal{O}_1 = l\big(\iota_I(x)\big) + l\big(\iota_I(y)\big) + l(w_{0,I}) - l(z) + \dim B_I.$$

By the proof of (i) we have

$$\dim \mathcal{O} = \dim(B \times B^-).\big(v w_{0,I}, \iota_I(y)\big).h_I.$$

The formula of (ii) follows from the previous formula and (2).

By Lemma 1.2 we have $\sigma.\mathcal{O} = (B \times B)(w w_{0,D}, x w_{0,D}).h_{I^*}$. Now

$$w w_{0,D} = w_{0,D} w^* = \iota_{I^*}(y^*) w_{0,I^*} z^*,$$

and similarly for $xw_{0,D}$. The formulas imply (iii), using that (u^*, u^*) fixes h_{I^*} if $u \in W_I$.

Finally, (iv) follows from the description of the isotropy group of h_I in $G \times G$ which was given above. □

Let V be the set of $B \times B$-orbits in X. For $v \in V$ we write $d(v) = \dim v$. We denote the orbit of (3) by $[I, x, w]$ or $[I, x, w]_G$. Thus, the elements of V are parametrized by triples $I \subset D$, $x \in W^I$, and $w = yz \in W$ (with $y \in W^I$, $z \in W_I$). By Lemma 1.3(ii)

$$\dim[I, x, w] = l(w_{0,D}) - l(x) + l(w) + |I|. \tag{5}$$

It follows from Lemma 1.3(iii) that

$$\sigma.[I, x, yz] = \left[I^*, \iota_{I^*}(y^*), \iota_{I^*}(x^*)w_{0,I^*}(z^*)^{-1}w_{0,I^*}\right].$$

For $I = D$, $[D, w, 1]$ is the double coset BwB. Its closure in X is the *large Schubert variety* S_w.

The combinatorial setup introduced in [RS] carries over—at least partly—to V and the subsets $V_I \subset V$ of $B \times B$-orbits in X_I ($I \subset D$).

Let M be the monoid $M(W \times W)$ (see [RS, 3.10]). It operates on V. Let $t = (s, 1)$ or $(1, s)$ be a simple reflection of $W \times W$ and put $P_s = B \cup BsB$ (a minimal parabolic subgroup of G). If $v \in V$ then $m(t).v$ is the open $B \times B$-orbit in $(P_s \times \{1\}).v$ if $t = (s, 1)$, and similarly for $t = (1, s)$. This defines an action of M on V, stabilizing all $V_I (I \subset D)$. If $m(t).v \neq v$ then $d(m(t).v) = d(v) + 1$ (cf. [RS, 7.2]).

In [MS, 4.1] an analysis is made of the action of a minimal parabolic group on the orbits of a Borel group in a spherical variety. This applies to the present situation, for the group $\boldsymbol{G} = G \times G$ and its spherical variety X. We use obvious notations like $\boldsymbol{B} = B \times B$, etc.

In general there are four possible cases, labeled I, II, III, IV in [MS, 4.1]. However, in the present case the situation is rather simple, as follows from the next lemma.

Recall that if $x \in W^I$ and $s \in S$ there are three possibilities:

(A) $sx > x$ and $sx \in W^I$;
(B) $sx > x$ and $sx = xt$ with $t \in S_I$;
(C) $sx < x$ in which case $sx \in W^I$.

1.4. Lemma. *Let $v \in V$ and let $\sigma \in \boldsymbol{S}$ be a simple reflection of $\boldsymbol{W}$.*

(i) *$\boldsymbol{P}_\sigma.v$ is the union of two $\boldsymbol{B}$-orbits.*
(ii) *If $m(\sigma).v \neq v$ then $\boldsymbol{P}_\sigma.v = v \cup m(\sigma).v$.*
(iii) *In the situation of* (ii) *the action map induces an isomorphism of the fibre product $(\boldsymbol{P}_\sigma - \boldsymbol{B}) \times_{\boldsymbol{B}} v$ onto $m(\sigma).v$.*

Proof. Let $v=[I,x,w]$ and put $w=yz$, where $y\in W^I$, $z\in W_I$. Then (cf. [B1, p. 151])

$$\begin{aligned} v &= (B\times B).(x,y).(B_I\times B_I)\bigl(1,z^{-1}\bigr).h_I \\ &= (B\times B).(x,y).(B\times B)\bigl(1,z^{-1}\bigr).h_I. \end{aligned}$$

Assume that $\sigma=(s,1) (s\in S)$. Then

$$\boldsymbol{P}_\sigma.v=(P_s\times B).(x,y).(B\times B)\bigl(1,z^{-1}\bigr).h_I.$$

By familiar Tits system properties $P_s.BxB=BxB\cup BsxB$. It follows that $\boldsymbol{P}_\sigma.v=v\cup v'$, where

$$v'=(B\times B).(sx,y).(B\times B)\bigl(1,z^{-1}\bigr).h_I.$$

In the cases (A) and (C) for x and s we have $v'=[I,sx,w]$. In case (B)

$$v\cup v'=(B\times B).(x,y).(P_t\times B_I).\bigl(1,z^{-1}\bigr).h_I=v\cup[I,x,sw],$$

and (i) follows. (iii) is also a consequence of these arguments and (ii) follows from the definition of $m(\sigma).v$.

We have proved the lemma for $\sigma=(s,1)$. For $\sigma=(1,s)$ $(s\in S)$ the proof is similar.

From (iii) it follows that v and σ are in the case II of [MS, 4.1.4]. □

1.5. Lemma. *Let $s\in S$ and $v=[I,x,w]\in V_I$.*

(i) *$m((s,1)).v\neq v$ if and only if we have for x and s case* (C) *or* (B) *with $wt>t$. In these cases we have, respectively, $m((s,1)).v=[I,sx,w]$ and $m((s,1)).v=[I,x,wt]$.*
(ii) *$m((1,s)).v\neq v$ if and only if $sw>w$, in which case $m((1,s)).v=[I,x,sw]$.*

Proof. This is a consequence of the proof of Lemma 1.4. □

Next we describe, following [B2, 3.1], a transversal slice Σ in $y=(\dot{x},\dot{w}).h_I$ to the $B\times B$-orbit $[I,x,w]$. Recall that this means (see [MS, 2.3.2]) that Σ is a locally closed subvariety of X containing y, of dimension

$$\dim X-\dim[I,x,w]=l(w_{0,D})+l(x)-l(w)+|D-I|,$$

such that the action map defines a smooth morphism $G\times\Sigma\to X$.

The closure $\overline{T}$ of T in X is a smooth toric variety, containing h_I. Let

$$\Sigma_I=\bigl\{z\in\overline{T}\bigm| h_I\in\overline{C_I.z}\bigr\}.$$

Then Σ_I is a transversal slice to $T.h_I$ in $\overline{T}$, isomorphic to affine space of dimension $|D-I|$.

Put $U=R_u(B)$, $U^-=R_u(B^-)$ and let ϕ be the morphism

$$\bigl(U^-\cap xUx^{-1}\bigr)\times\bigl(U^-\cap wU^-w^{-1}\bigr)\times\Sigma_I\to X$$

sending (g,h,z) to $(g\dot{x},h\dot{w}).z$.

1.6. Proposition.
(i) *The image Σ of ϕ is a transversal slice in y to $[I, x, w]$.*
(ii) *There is a cocharacter of T which contracts S to y.*

Proof. This is a version of [B2, Theorem 3.1]. The proof is as in [B2]. It also follows that Σ is an attractive slice in the sense of [B2], which implies the property of (ii) (defined in [MS, 2.3.2]). $\square$

We end this section with some facts on local systems on the $(B \times B)$-orbits, needed in Section 5. If A is a torus denote by $X(A)$ its character group and by $\widehat{X}(A)$ the tensor product $X(A) \otimes_{\mathbb{Z}} (\mathbb{Z}_{(p)}/\mathbb{Z})$, where $\mathbb{Z}_{(p)}$ is the localization at the prime ideal (p) of $\mathbb{Z}$, p being the characteristic. The group $\widehat{X}(A)$ parametrizes the (tame) local systems on A which have rank one, see [MS, 2.1].

If $Y(A)$ is the group of cocharacters of A we have, similarly, a group $\widehat{Y}(A)$. Between $\widehat{X}(A)$ and $\widehat{Y}(A)$ there is a pairing $\langle\, ,\rangle$, with values in $\mathbb{Z}_{(p)}/\mathbb{Z}$.

Since G is adjoint the character group $X(T)$ has basis D. Likewise, $X(T_I)$ has basis I, whence an injection $X(T_I) \to X(T)$, which is the homomorphism induced by $T \to T_I$. Clearly, $X(T_I)$ is a direct summand of $X(T)$, from which we see that the induced homomorphism $\widehat{X}(T_I) \to \widehat{X}(T)$ is injective (cf. [MS, 2.1.5], recall that the kernel C_I of the homomorphism $T \to T_I$ is connected).

We shall identify $\widehat{X}(T_I)$ with a subgroup of $\widehat{X} = \widehat{X}(T)$. In the sequel we shall write $\widehat{X} = \widehat{X}(T)$ and $\widehat{X}_I = \widehat{X}(T_I)$.

1.7. Let $v \in V$. If $\dot{v} \in v$ the unipotent radical of the isotropy group $(B \times B)_{\dot{v}}$ is connected by Lemma 1.3(iv). The projection of $(B \times B)_{\dot{v}}$ onto $T \times T$ is independent of the choice of $\dot{v}$ by [MS, 2.2.5]. We denote it by $(T \times T)_v$. Denote by ϕ_v the homomorphism $T \times T \to T_I$ sending $(x(t), w(t'))$ to $t(t')^{-1}C_I$. Then ϕ_v induces an isomorphism $(T \times T)/(T \times T)_v \simeq T_I$ by Lemma 1.3(iv).

By [MS, 2.2.3] the rank one local systems on v which have a weight for the $B \times B$-action are of the form $\widehat{\phi_v}\xi$ $(\xi \in \widehat{X}(T_I))$, where $\widehat{\phi_v}$ is the induced homomorphism $\widehat{X}(T_I) \to \widehat{X}(T \times T)$. Let λ_v (ρ_v) be the composite of ϕ_v and the injection $t \mapsto (t, 1)$ (respectively $t \mapsto (1, t)$) of T into $T \times T$.

1.8. Lemma. *Let $v = [I, x, w], \xi \in \widehat{X}_I$. Then $\widehat{\lambda_v}(\xi) = x^{-1}.\xi$, $\widehat{\rho_v}(\xi) = -w^{-1}.\xi$, and $\widehat{\phi_v} = (\widehat{\lambda_v}, \widehat{\rho_v})$.*

Proof. The proof is straightforward. $\square$

2. The Bruhat order

On V and its subsets V_I we have a “Bruhat order,” $\leqslant$, defined by the inclusion of orbit closures. We discuss it in this section. The order on W and its subsets W_I and W^I is the usual one.

2.1. Lemma. *Let $v, v' \in V$ and let $t \in \boldsymbol{S}$. Then*

(a) $v \leqslant m(t).v$;
(b) *if $v' \leqslant v$ then $m(t).v' \leqslant m(t).v$;*
(c) *if $v' \leqslant v$ then $d(v') \leqslant d(v)$, with equality if and only if $v = v'$;*
(d) *$V_I (I \subset D)$ has a unique element of minimal dimension, viz. $B_I = [I, w_{0,D} w_{0,I}, 1]$.*

Proof. The last point follows by using the dimension formula (5). (It will follow from the next lemma that B_I is also the unique minimal element of the Bruhat order.) The other points are proved as similar results in [RS, 7.2]. $\square$

From Lemma 1.5 we see that V_I is an M-set whose order is compatible with the M-action in the sense of [RS, Section 5].

2.2. Lemma. *Let $I \subset D, x, x' \in W^I$, $w, w' \in W$. Then $[I, x', w'] \leqslant [I, x, w]$ if and only if there exists $u \in W_I$ such that $xu^{-1} \leqslant x'$, $w'u \leqslant w$.*

Proof. We have

$$\dim[I, x, w] - \dim B_I = l(x w_{0,I} w_{0,D}) + l(w).$$

Let $(t_1, \ldots, t_l)$ be a reduced decomposition of $(x w_{0,I} w_{0,D}, w) \in \boldsymbol{W}$, the t_i being simple reflections of $\boldsymbol{W}$. It follows, by repeated application of Lemma 1.5(ii), that

$$[I, x, w] = m(t_1) \ldots m(t_l).B_I = m\big((x w_{0,I} w_{0,D}, w)\big).B_I.$$

By familiar arguments (cf. [BT, 3.13]) one shows that

$$\overline{[I, x, w]} = \boldsymbol{P}_{t_1} \ldots \boldsymbol{P}_{t_l}.B_I,$$

from which one concludes that $\overline{[I, x, w]}$ is the union of the orbits $m((c, d)).B_I$, where $c \leqslant x w_{0,I} w_{0,D}$ and $d \leqslant w$.

Let $[I, x', w'] \leqslant [I, x, w]$ and take c, d as before such that $[I, x', w'] = m((c, d)).B_I$. Write $c = c' u w_{0,I} w_{0,D}$, with $c' \in W^I, u \in W_I$. Then $c' u w_{0,I} \geqslant x w_{0,I}$ and

$$\begin{aligned}[I, x', w'] &= m\big((c, d)\big).B_I = m\big((c' u w_{0,I} w_{0,D}, d)\big).B_I \\ &= (B \times B).(c'u, d).h_I = \big[I, c', du^{-1}\big].\end{aligned}$$

We conclude that $x' = c'$, $w' = du^{-1}$. Hence $w'u \leqslant w$ and $xw_{0,I} \leqslant x'uw_{0,I}$. It follows that there exist $u_1, u_2 \in W_I$ such that

$$xu_1 \leqslant x', \qquad u_2 \leqslant uw_{0,I}, \qquad u_1u_2 = w_{0,I}.$$

Then

$$xu^{-1} \leqslant xw_{0,I}u_2^{-1} = xu_1 \leqslant x'.$$

So there exists u with the asserted properties. Conversely, if so, then

$$xw_{0,I} = xu^{-1}uw_{0,I} \leqslant x'uw_{0,I},$$

and we conclude that there exist c and d as before, whence $[I, x', w'] \leqslant [I, x, w]$. $\square$

Let J be a subset of D containing I. Let $x, y \in W^J$, $z \in W_J$, $w = yz$.

2.3. Lemma. *If $\overline{[J, x, w]} \cap \overline{X_I} \neq \emptyset$ then $I \subseteq J$. If this is so the intersection is the union of the orbit closures $\overline{[I, xv, wv]}$ with $v \in W_J \cap W^I$ and $l(wv) = l(w) + l(v)$.*

Proof. The first point is immediate. By [B1, 2.1, Theorem] we have for $x, y \in W^J$, $z \in W_J$:

$$\overline{(B \times B^-).(xz^{-1}, y).h_J} \cap \overline{X_I} = \bigcup \overline{(B \times B^-).(xz^{-1}v, yv).h_I},$$

where v runs through the elements of W_J such that $yv \in W^I$ and $l(xz^{-1}) = l(xz^{-1}v) + l(v)$. Such a v lies in $W_J \cap W^I$, from which one sees also that $l(yvu) = l(y) + l(vu)$ for $u \in W_I$.

Since $x \in W^J$, we have $l(z) = l(z^{-1}v) + l(v)$. Write $xz^{-1}v = x_1u_1$, with $x_1 \in W^I$, $u_1 \in W_I$ and put $v_1 = z^{-1}vu_1^{-1}$. Then $xv_1 \in W^I$ and $l(yzv_1) = l(yvu_1^{-1}) = l(y) + l(vu_1^{-1})$. We can conclude that

$$\overline{(B \times B^-).(x, w).h_J} \cap \overline{X_I} = \bigcup \overline{(B \times B^-).(xv, wv).h_I},$$

where v runs through the elements of W_J with $xv \in W^I$, i.e. $v \in W_J \cap W^I$ and $l(w) = l(wv) + l(v)$. Using the relation between $B \times B^-$-orbits and $B \times B$-orbits of the proof of Lemma 1.2 we obtain the assertion of the lemma. $\square$

2.4. Proposition. *Let $x' \in W^I$, $x \in W^J$, $w, w' \in W$. Then $[I, x', w'] \leqslant [J, x, w]$ if and only if $I \subseteq J$ and there exist $u \in W_I$, $v \in W_J \cap W^I$ with $xvu^{-1} \leqslant x'$, $w'u \leqslant wv$, and $l(wv) = l(w) + l(v)$. If this is so we have $x' \geqslant x$ and $-l(x') + l(w') \leqslant -l(x) + l(w)$.*

Proof. The preceding lemmas imply the first assertions. If u and v with the asserted properties exist then, since $x \in W^J$,

$$x' \geqslant xvu^{-1} \geqslant x.$$

The last inequality follows from the fact that $\boldsymbol{B}$-orbit closures in $\overline{X_J}$ intersect $\overline{X_I}$ properly if $I \subset J$ (see [B2, 1.4]) and the dimension formula (ii) from Lemma 1.3. □

2.5. Corollary.
(i) $[I, x', w'] \leqslant [J, x, 1]$ *if and only if* $I \subseteq J$, $w' \in W_J$ *and* $xw' \leqslant x'$.
(ii) $[I, x', w'] \subseteq [D, 1, w]$ *if and only if there exists* $w_1 \leqslant w$ *with* $w_1^{-1}w' \leqslant x'$.
(iii) $[I, x, w] \subseteq \overline{B}$ *if and only if* $w \leqslant x$.

Proof. By the proposition, $[I, x', w'] \leqslant [J, x, 1]$ if and only if $I \subseteq J$ and there exist $u \in W_I, v \in W_J \cap W^I$ with $xvu^{-1} \leqslant x'$ and $w'u \leqslant v$. From the second relation we infer that $w' \leqslant vu^{-1}$. Since $vu^{-1} \in W_J$, w' also lies in W_J. Moreover, $xw' \leqslant xvu^{-1} \leqslant x'$. We have established the conditions of (i).

If, conversely, they are satisfied, write $xw' = xvu^{-1}$ with $u \in W_I$, $xv \in W^I$. Then $w' = vu^{-1}$. Since $w' \in W_J$ the same holds for v, so $v \in W_J \cap W^I$. It follows that u and v are as required in the proposition, and $[I, x', w'] \leqslant [J, x, 1]$. This proves (i). The special case $J = D$ gives (iii).

By the proposition, $[I, x', w'] \leqslant [J, 1, w]$ if and only if $I \subseteq J$ and there exist $u \in W_I, v \in W^J \cap W^I$ with $vu^{-1} \leqslant x'$ and $w'u \leqslant wv$, $l(wv) = l(w) + l(v)$. There exist $w_1 \leqslant w$, $v_1 \leqslant v$ such that $w'u = w_1v_1$ and $l(w_1v_1) = l(w_1) + l(v_1)$. Then $w_1^{-1}w' = v_1u^{-1} \leqslant vu^{-1} \leqslant x'$, establishing the condition of (ii).

To prove the sufficiency of the condition it suffices to deal with the case that $w_1 = w$. So $w^{-1}w' \leqslant x'$. Write $w^{-1}w' = vu^{-1}$, with $v \in W^I$, $u \in W_I$. Then $w'u = wv$. There exist $w_1 \leqslant w$ and $v_1u_1 \leqslant v$ ($v_1 \in W^I, u_1 \in W_I$) such that $w'u = w_1v_1u_1$ and $l(w_1v_1u_1) = l(w_1) + l(v_1) + l(u_1)$. Then $w'w_1v_1uu_1^{-1} \leqslant w_1v_1$, $v_1u_1u^{-1} \leqslant vu^{-1} \leqslant x'$, and $l(w_1v_1) = l(w_1) + l(v_1)$. It follows that $[I, x', w'] \leqslant [D, 1, w_1] \leqslant [D, 1, w]$. We have proved (ii). (iii) is also a special case of (ii). □

The explicit description of the Bruhat order of V given in Proposition 2.4 is a bit cumbersome. We present another description which is somewhat more transparent.

Define two relations $\leqslant_1$ and $\leqslant_2$ on V by $[I, x', w'] \leqslant_1 [J, x, w]$ if $I \subseteq J$ and $x' \geqslant x$, $w' \leqslant w$, and $[I, x', w'] \leqslant_2 [J, x, w]$ if $I \subseteq J$ and there is $z \in W_J$ with $xz \leqslant x'$ and $w' = wz$, $l(wz) = l(w) + l(z)$.

2.6. Lemma.
(i) *If* $v, v' \in V$, $v' \leqslant_i v$ $(i = 1, 2)$ *then* $v' \leqslant v$.
(ii) $\leqslant_1$ *and* $\leqslant_2$ *are order relations.*

Proof. (i) is obvious for $\leqslant_1$ (take $u = v = 1$ in Proposition 2.4). If z is as in the definition of $\leqslant_2$, write $z = vu^{-1}$ $(v \in W^I, u \in W_I)$. Then u and v are as required in Proposition 2.4.

That $\leqslant_1$ is an order is obvious. The proof that $\leqslant_2$ is an order is straightforward. $\square$

2.7. Lemma. *Let $v = [J, x, w]$, $v' = [I, x', w']$, and assume that $v' \leqslant v$. There is $\tilde{v} \in V$ with $v' \leqslant_2 \tilde{v} \leqslant_1 v$.*

Proof. By Proposition 2.4 there is $w_1 \leqslant w$ and $v_1 u_1 \leqslant v$ ($v_1 \in W^I$, $u_1 \in W_I$) with $w'u = w_1 v_1 u_1$, $l(w_1 v_1 u_1) = l(w_1) + l(v_1) + l(u_1)$. Then $w'uu_1^{-1} = w_1 v_1$, $l(w_1 v_1) = l(w_1) + l(v_1)$, and $xvu_1u^{-1} \leqslant vu^{-1} \leqslant x'$. It follows that $v' \leqslant_2 [J, x, w_1]$. Moreover, $[J, x, w_1] \leqslant_1 v$. Hence $\tilde{v} = [J, x, w_1]$ is as required. $\square$

2.8. Proposition. *$\leqslant$ is the order generated by $\leqslant_1$ and $\leqslant_2$.*

Proof. By definition, the order on V generated by $\leqslant_1$ and $\leqslant_2$ is such that v' is majorized by v for that order if and only if there exists a chain $v_0 = v'$, $v_1, \ldots, v_s = v$ of elements of V such that for $i = 1, \ldots, s$ either $v_{i-1} \leqslant_1 v_i$ or $v_{i-1} \leqslant_2 v_i$. By the preceding lemmas it is immediate that this order is $\leqslant$. $\square$

In Section 4 we shall find the Möbius function of our order, see the remark after Proposition 4.6.

If $x \in W$ put $I(x) = \{\alpha \in D \mid x.\alpha \in R^+\}$ and for $x, w \in W$ put

$$X_{x,w} = \bigcup_{I \subseteq I(x)} [I, x, w].$$

2.9. Lemma.

(i) $[I, x, w] \leqslant [I(x), x, w]$;

(ii) $\overline{X_{x,w}} = \overline{[I(x), x, w]}$;

(iii) *A large Schubert variety $S_w = \overline{X_{D,1,w}}$ is a disjoint union of sets $X_{x',w'}$.*

Proof. It is immediate that $[I, x, w] \leqslant_1 [I(x), x, w]$. (i) then follows from Proposition 2.8, (ii) is a consequence of (i), and (iii) follows from Corollary 2.5(ii). $\square$

2.10. Proposition. *$X_{x,w}$ is a locally closed subvariety of X isomorphic to affine space of dimension $l(w_{0,D}) - l(x) + l(w) + |I(x)|$.*

Proof. For $\alpha \in R$ let U_α be the one-parameter additive subgroup of G defined by α. If $y \in W$ let U_y be the subgroup generated by the U_α with $\alpha \in R^+$, $y^{-1}.\alpha \in -R^+$. It is a subgroup of the unipotent part U of B.

We have $[I, x, w] = (B \times U).(\dot{x}, \dot{w}).h_I$. The isotropy group K of $(\dot{x}, \dot{w})$ in $B \times U$ is the set of $(tu, u') \in B \times U$ with

$$\left(\dot{x}^{-1} t u \dot{x}, \dot{w}^{-1} u' \dot{w}\right) = (clv, lv'),$$

where $c \in C_I$, $l \in L_I$, $v \in R_u(P_I^-)$, $v' \in R_u(P_I)$ (see [B1, Proposition A1]).

The morphism

$$(u, u', tC_I) \mapsto (u, u').(\dot{x}, \dot{w}).t.h_I$$

of $U_{xw_{0,D}} \times U_w \times T/C_I$ to $[I, x, w]$ is bijective. This follows from the observation that $(U_{xw_{0,D}} \times U_w) \cap K = \{1\}$.

We infer that

$$\begin{aligned}(\dot{x}^{-1}, \dot{w}^{-1}).X_{x,w} &= (\dot{x}^{-1}U_{xw_{0,D}}\dot{x}, \dot{w}^{-1}U_w\dot{w}).\Big(\bigcup_{I \subseteq I(x)} T.h_I\Big)\\ &\subset (U \times U^-).\overline{T.h_D}.\end{aligned}$$

Using the results of [DS, 3.7–3.8], we see that

$$(\dot{x}^{-1}U_{xw_{0,D}}\dot{x}, \dot{w}^{-1}U_w\dot{w}).\Big(\bigcup_{I \subseteq I(x)} T.h_I\Big)$$

is a closed subspace of $(U \times U^-).\overline{T.h_D}$ isomorphic to an affine space. Using (5) we see that its dimension is as stated. Observing that $(U \times U^-).\overline{T.h_D}$ is open in X, the proposition follows. □

2.11. From the description of $X_{x,w}$ given in the proof of Proposition 2.10, it is immediate that there is a cocharacter of $T \times T$, independent of x, w, which contracts $X_{x,w}$ to the fixed point $(xB^-, wB) \in X_\emptyset$ of $T \times T$ in X. Hence the cellular decomposition of X is a Byałinicki–Birula decomposition of the smooth variety X. (Our decomposition is closely related to the Byałinicki–Birula decomposition of [B1, 3.3].)

It is known that the union of the cells of dimension $\leqslant i$ is closed in X. It follows that a large Schubert variety S_w, which is a union of cells by Lemma 2.9(iii), enjoys the same property. It is well known that this implies that the odd cohomology of S_w vanishes and that its $2i$th Betti number equals the number of i-dimensional cells. This leads to the following result. If X is an algebraic variety, we denote by

$$\mathcal{P}_X(t) = \sum_{i \geqslant 0} \dim H^i(X)t^i$$

its Poincaré polynomial, with constant coefficients (in l-adic cohomology, or in classical cohomology if $k = \mathbb{C}$).

2.12. Corollary. *The Poincaré polynomial $\mathcal{P}_{S_w}$ equals*

$$\sum_{[I(a),a,b] \leqslant [D,1,w]} t^{2(l(w_{0,D}) - l(a) + l(b) + |I(a)|)}.$$

Corollary 2.5(ii) makes the summation more explicit.

In the particular case $w = 1$, we have $S_1 = \overline{B}$. Then the formula simplifies by Corollary 2.5(iii) to

$$\mathcal{P}_{\overline{B}}(t) = \sum_{b \leqslant a} t^{2(l(w_{0,D})-l(a)+l(b)+|I(a)|)}.$$

For $G = PGL_2$ the right-hand side is $1 + t^2 + t^4$ and for $G = PGL_3$ it is $1 + 2t^2 + 4t^4 + 7t^6 + 4t^8 + t^{10}$.

In the particular case $w = w_{0,D}$, one obtains a known formula for $\mathcal{P}_X$ (see [DP, 7.7]).

Another consequence of Proposition 2.10 is that the Chow group $A_*(X)$ is freely generated by the classes $[\overline{X_{x,w}}]$, which is a reformulation of a result due to Brion (see [B1, 3.3]).

3. A Hecke algebra representation

3.1. Let $\mathcal{H}$ be the Hecke algebra of W. It is a free module over the ring of Laurent polynomials $\mathbb{Z}[u, u^{-1}]$, with a basis $(e_w)_{w \in W}$. The multiplication is defined by the rules

$$e_s e_w = \begin{cases} e_{sw} & \text{if } sw > w, \\ (u^2 - 1)e_w + u^2 e_{sw} & \text{if } sw < w, \end{cases}$$

where $w \in W$, $s \in S$.

The variety X is a spherical variety for $\boldsymbol{G} = G \times G$. We now invoke the results of [MS], where for any spherical variety a module $\mathcal{M}$ over a Hecke algebra is constructed (on the model of the work of Lusztig and Vogan in [LV] in the case of symmetric varieties). In our case this is the Hecke algebra associated to $\boldsymbol{W}$, i.e. $\mathcal{H} \otimes_{\mathbb{Z}[u,u^{-1}]} \mathcal{H}$.

There are several technicalities which have to be taken care of. In the first place, one takes the base field k to be an algebraic closure of a finite field $\boldsymbol{F}_q$, and assumes all ingredients of the constructions to be defined over $\boldsymbol{F}_q$ (which is possible, as there are only finitely many such ingredients). The module $\mathcal{M}$ is free, with a basis indexed by the set V of orbits v of $\boldsymbol{B}$ in X. In the general situation considered in [MS], the basis elements also involve local systems on the orbits. In the present section we consider the case that all local systems are trivial (that this is possible is a consequence of the fact that, with the notations of [MS, 4.1.4], in X only the case II occurs). A more general situation, where non-trivial local systems on the orbits are taken into account, will be taken up in Section 5.

In the setup of [MS] a basis element m_v defined by $v \in V$ comes as a class in a Grothendieck group. More precisely, let $\mathcal{A}_X$ be the category of constructible $\overline{\mathbb{Q}_l}$-sheaves $\mathcal{S}$ on X, provided with an isomorphism $\Phi : F^*\mathcal{S} \to \mathcal{S}$ (where F is the Frobenius morphism). $(\mathcal{S}, \Phi)$ and $(\mathcal{S}, \Phi')$ are identified if $\Phi^n = (\Phi')^n$ for some n. Some further conditions are imposed, which need not be spelled out. The pairs are the objects of an abelian category $\mathcal{A}_X$, whose Grothendieck group is denoted by $\mathcal{K}(\mathcal{A}_X)$.

Put $E = \overline{\mathbb{Q}_l}$ and denote for $v \in V$ by E_v the sheaf on X which restricts to the constant sheaf E on v and to 0 on the complement of v. Now $m_v \in \mathcal{K}(\mathcal{A}_X)$ is the class of (E_v, ϕ), where ϕ is the Frobenius map (m_v corresponds to $\epsilon_{0,v}$ of [MS, 4.3] and e_s to $\epsilon_{0,s}$).

Another technicality is that the base ring $\mathbb{Z}[u, u^{-1}]$ has to be extended (provisionally) to a ring $\mathcal{R}$, the group ring $\mathbb{Z}[C]$ of a group C deduced from the eigenvalues of Frobenius endomorphisms acting on the stalks of certain sheaves. C is a subgroup of the multiplicative group of non-zero algebraic numbers modulo roots-of-unity. Then $\mathbb{Z}[u, u^{-1}]$ is the group ring of the group generated by the image u of $q^{1/2}$.

To *define* the Hecke algebra action the extension to $\mathcal{R}$ is not needed. But if one is after more delicate properties of $\mathcal{M}$, such as the existence of a Kazhdan–Lusztig basis, the introduction of the ring extension can not be avoided.

If $v = [I, x, w]$ (as before) we write $m_v = m_{I,x,w}$. The next lemma describes the $\mathcal{H} \otimes \mathcal{H}$-action on $\mathcal{M}$. In (i) we have the three cases (A)–(C) for x and s; see Lemma 1.5. In case (B) we put $sx = x\sigma$.

3.2. Lemma. *Let $x \in W^I$, $w \in W$, $t \in \boldsymbol{S}$.*

(i) *If $t = (s, 1)$, $e_t.m_{I,x,w}$ equals*
   - (a) $(u^2 - 1)m_{I,x,w} + u^2 m_{I,sx,w}$ *in case* (A);
   - (b1) $m_{I,x,w\sigma}$ *in case* (B) *if* $w\sigma > w$;
   - (b2) $(u^2 - 1)m_{I,x,w} + u^2 m_{I,x,w\sigma}$ *in case* (B) *if* $w\sigma < w$;
   - (c) $m_{I,sx,w}$ *in case* (C).

(ii) *If $t = (1, s)$, $e_t.m_{I,x,w}$ equals either $m_{I,x,sw}$ if $sw > w$, or $(u^2 - 1)m_{I,x,w} + u^2 m_{I,x,sw}$ if $sw < w$.*

Proof. The formulas are proved as in [MS, 4.3.4, 4.3.9], taking into account Lemmas 1.4(iii) and 1.5. □

3.3. There is an action of $\boldsymbol{W}$ on V (see [K]). Since only case II occurs the action can easily be described (for example, using [MS, Remark, p. 78]). Notice that if $t \in \boldsymbol{S}$ and $m(t).v \neq v$ we have $t.v = m(t).v$.

Explicitly, the action is given by: $(s, 1).[I, x, w]$ equals $[I, sx, w]$ in the cases (A) and (C) and $[I, x, w\sigma]$ in case (B) (notations being as in Lemma 3.2). Also, $(1, s).[I, x, w] = [I, x, sw]$ in all cases.

Notice that the formulas of Lemma 3.2 can be rewritten as

$$e_t.m_v = \begin{cases} m_{t.v} & \text{if } d(t.v) > d(v), \\ \left(u^2 - 1\right)m_v + u^2 m_{t.v} & \text{if } d(t.v) < d(v). \end{cases} \tag{6}$$

The construction of our representation given in [MS] is non-elementary, it uses l-adic sheaves. One can verify in a more elementary way that the formulas of the proposition define a representation of $\mathcal{H} \otimes \mathcal{H}$ (see Section 6.1).

But we now shall need the sheaf theoretical approach. Verdier duality theory leads to an involutorial map Δ of $\mathcal{M}$, which is semilinear in $\mathcal{R}$ relative to the involution defined by the inverse in the group C and satisfies

$$\Delta(e_t.m) = e_t^{-1}.\Delta(m) \quad (t \in \boldsymbol{S},\ m \in \mathcal{M}), \tag{7}$$

(see [MS, 3.3.2, 4.4.7], where Δ is denoted by D).

3.4. Lemma.
(i) *There exist elements $b_{w,v} \in \mathcal{R}$ such that*

$$\Delta(m_v) = u^{-2d(v)} \sum_{w \in V} b_{w,v} m_w.$$

(ii) $b_{w,v} \neq 0$ *if and only if* $w \leqslant v$ *and* $b_{v,v} = 1$.

Proof. A formula similar to (i) is in [MS, 3.4]. However, in that formula other terms could appear, corresponding to non-constant local systems on the orbits w. But by the last line of [MS, 3.4.1] only the constant local system on x will appear, since in our situation all maps $\widehat{\phi_v}$ are injective (as a consequence of Lemma 1.8).

For the proof of (ii) we have to go into the definition of Δ. Denote by $\Delta(E_v)$ the Verdier dual of the sheaf E_v, an object in a derived category. By [MS, 3.3],

$$b_{w,v} = \sum_i (-1)^i \left(\sum_{\rho_i} m_{\rho_i,i} \rho_i \right), \tag{8}$$

where ρ_i runs through the images in C of the eigenvalues of the Frobenius map of the stalk $\mathcal{H}^i(\Delta(E_v))_a$ of the cohomology sheaf $\mathcal{H}^i(\Delta(E_v))$ in a point $a \in v(\boldsymbol{F}_q)$, the $m_{\rho_i,i}$ denoting multiplicities.

By general facts about Verdier duality, $\mathcal{H}^i(\Delta(E_v))_a$ is the dual of the local cohomology group $H_{[a]}^{-i}(X, E_v)$. By Proposition 1.6(i), there exists a transversal slice S at a to the orbit w. Then, locally in a for the étale topology, X is the product of w and S. Hence

$$\mathcal{H}_{[a]}^i(X, E_v) = \mathcal{H}_{[a]}^{i-2d(w)}(S, E_v).$$

By Proposition 1.6(ii) there is a cocharacter of T contracting S to a. Then by [MS, Remark after 2.3.1] $H_{[a]}^i(S, E_v)$ is isomorphic to the cohomology group with proper support $H_c^i(S, E_v) = H_c^i(S \cap v, E)$. Now $S \cap v \neq \emptyset$ if and only if $w \subset \bar{v}$. If this is so, it follows from (8) that $b_{w,v} \neq 0$. In fact, up to a power of q the right-hand side of (8) equals the number of $\boldsymbol{F}_q$-rational points of $S \cap v$ which is $\neq 0$ (after enlarging $\boldsymbol{F}_q$, if necessary).

We have shown that $b_{w,v} \neq 0$ if $w \leqslant v$. The converse follows from the fact that the dual $\Delta(E_v)$ is zero outside $\bar{v}$. Finally, $b_{v,v} = 1$ follows from the fact that $\bar{v}$ is smooth in the points of v. □

3.5. Proposition.
(i) *If $b_{w,v} \neq 0$, it is a polynomial in $\mathbb{Z}[u^2]$ with leading term $(-u^2)^{d(v)-d(w)}$.*
(ii) $(-u^2)^{d(v)-d(w)} b_{w,v}(u^{-2}) = b_{w,v}(u^2)$.

Proof. Let $v \in V$ and assume that there is $t \in \boldsymbol{S}$ such that $d(t.v) < d(v)$, whence by (7)

$$\Delta(m_v) = e_t^{-1}.\Delta(m_{t.v}).$$

Writing this out in terms of the b's and using

$$e_t^{-1} = u^{-2}\bigl(e_t - u^2 + 1\bigr),$$

we obtain for $d(t.v) < d(v)$:

$$b_{w,v} = \begin{cases} b_{t.w,t.v} & \text{if } d(t.w) < d(w), \\ \bigl(1-u^2\bigr) b_{w,t.v} + u^2 b_{t.w,t.v} & \text{if } d(t.w) > d(w). \end{cases} \tag{9}$$

Using these formulas, a straightforward induction shows that the proof of part (i) is reduced to the case $d(t.v) \geqslant d(v)$ for all $t \in \boldsymbol{S}$. Then v is of the form $[J, w_{0,D}w_{0,J}, 1]$ by Lemma 2.1.

Put $\tilde{b}_{w,v} = u^{-d(v)+d(w)} b_{w,v}$. Then (9) shows that

$$\tilde{b}_{w,v} = \begin{cases} \tilde{b}_{t.w,t.v} & \text{if } d(t.w) < d(w), \\ \bigl(u^{-1}-u\bigr) \tilde{b}_{w,t.v} + \tilde{b}_{t.w,t.v} & \text{if } d(t.w) > d(w). \end{cases} \tag{10}$$

Using these formulas, by induction $\tilde{b}_{w,v}$ is a polynomial in $u^{-1} - u$, hence is invariant under the change $u \mapsto -u^{-1}$. Then (ii) follows from (i).

It remains to deal with the case $v = [J, w_{0,D}w_{0,J}, 1]$. In the sequel the R-polynomials of Kazhdan–Lusztig (see [KL, Section 2]) will appear. They lie in $\mathbb{Z}[u^2]$. They are defined in terms of the Hecke algebra $\mathcal{H}$ by

$$e_{x^{-1}}^{-1} = u^{-2l(x)} \sum_y (-1)^{l(x)-l(y)} R_{y,x}\bigl(u^2\bigr) e_y,$$

where $x, y \in W$. From (7) we deduce

$$b_{[D,1,y],[D,1,x]} = (-1)^{l(x)-l(y)} R_{y,x}.$$

We have $R_{y,x} = 0$ if $y \nleqslant x$ and $R_{x,x} = 1$. The R-polynomials satisfy the following recursive relations (where $x, y \in W$, $s \in S$). Together with the boundary conditions $R_{y,1} = \delta_{y,1}$, these relations define the R-polynomials uniquely.

$$R_{y,sx} = \begin{cases} R_{sy,x} & \text{if } sx > x,\ sy < y, \\ \bigl(u^2-1\bigr) R_{y,x} + u^2 R_{sy,x} & \text{if } sx > x,\ sy > y. \end{cases} \tag{11}$$

We have similar relations to the right action of a simple reflection. Moreover, if $R_{y,x} \neq 0$, it has leading term $u^{2(l(x)-l(y))}$.

We return to the determination of the $b_{w,v}$ for $v = [J, w_{0,D}w_{0,J}, 1]$. First let $J = D$. Then $v = [D, 1, 1]$ is the $\boldsymbol{B}$-orbit $B \subset G$. By Proposition 3.5(i) and Corollary 2.5(iii), $b_{w,v} \neq 0$ if and only if $w = [I, a, b]$, with $I \subseteq J$ and $b \leqslant a$. Write $\beta^I_{a,b} = b_{[I,a,b],[D,1,1]}$. $\square$

3.6. Lemma. $\beta^I_{a,b} = (-1)^{l(a)+l(b)}(1-u^2)^{|D-I|}R_{b,a}(u^2)$.

Proof. Let $s \in S$. Then

$$e_{(s,1)}.m_{D,1,1} = e_{(1,s)}.m_{D,1,1} = m_{D,1,s}.$$

This implies, using (7):

$$e_{(s,1)}.\Bigl(\sum \beta^I_{a,b}m_{I,a,b}\Bigr) = e_{(1,s)}.\Bigl(\sum \beta^I_{a,b}m_{I,a,b}\Bigr).$$

Fix $a \in W^I$ and assume $s \in S$ is such that $sa > a$. Using Lemma 3.2 we determine the coefficients of $m_{I,a,b}$ on both sides of the preceding equation. We obtain

$$\beta^I_{sa,b} = \begin{cases} \beta^I_{a,sb} & \text{if } sb < b, \\ (1-u^2)\beta^I_{a,b} + u^2\beta^I_{a,sb} & \text{if } sb > b. \end{cases}$$

Moreover, we see from Lemma 3.4(ii) that $\beta^I_{1,b} = \delta_{b.1}\beta^I_{1,1}$. Comparing the preceding formulas with the inductive formulas (11) for the R-polynomials we conclude that

$$\beta^I_{a,b} = (-1)^{l(a)+l(b)}R_{b,a}(u^2)\beta^I_{1,1}.$$

It remains to determine $\beta^I_{1,1} = b_{[I,1,1],[D,1,1]}$. Let $\overline{T}$ be the closure of T in X. It contains the point $h = h_I$ of $[I, 1, 1]$. We use again that by [B2, 3.1, Theorem] there exists a contractible transverse slice S at h to $[I, 1, 1]$. It follows from [B2] that S can be taken to be a transverse slice in $\overline{T}$ at h to $T_I.h$. From the proof of Lemma 3.4(ii) we see how to determine $\beta^I_{1,1}$: we have to study $S \cap B$ and the action of the Frobenius map on its cohomology. Now $S \cap B = S \cap T$ is a torus isomorphic to C_I. From familiar results about the Frobenius action on the cohomology of $\boldsymbol{F}_q$-split tori we then obtain that $\beta^I_{1,1} = (1-u^2)^{|D-I|}$, finishing the proof of Lemma 3.6. $\square$

Fix J and put $v_J = [J, w_{0,D}w_{0,J}, 1]$. Let $I \subseteq J$.

3.7. Lemma. *Let $a \in W^I$, $b \in W_J$, and $w_{0,D}w_{0,J}b \leqslant a$. Then*

$$b_{[I,a,b],v_J} = (-1)^{l(a)+l(b)+l(w_{0,D}w_{0,J})}\bigl(1-u^2\bigr)^{|J-I|}R_{w_{0,D}w_{0,J}b,a}.$$

Proof. By Corollary 2.5(i) we know that $b_{[I,a,b],v_J} \neq 0$ if and only if $I \subseteq J$, $b \in W_J$, and $w_{0,D}w_{0,J}b \leqslant a$. Write $a = a_1c$, with $a_1 \in W^J$, $c \in W_J$. The preceding inequality can only hold if $a_1 = w_{0,D}w_{0,J}$ and then $b \leqslant c$. Since $a \in W^I$, we have $c \in W_J \cap W^I$. By [KL, 2.1(iv)], applied for W and W_J,

$$R_{w_{0,D}w_{0,J}b,a} = R_{w_{0,D}w_{0,J}b,w_{0,D}w_{0,J}c} = R_{b,c}.$$

Now $[I, w_{0,D}w_{0,J}c, b]$ lies in $\overline{X_J}$ and is, in fact, the $B_J \times B_J$-orbit $[J, c, b]_{G_J}$. The statement of the lemma then asserts that

$$b_{[I,w_{0,D}w_{0,J}c,b]_G,[J,w_{0,D}w_{0,J},1]_G} = b_{[I,c,b]_{G_J},[J,1,1]_{G_J}}. \tag{12}$$

We have the Hecke algebra $\mathcal{H}_J$ of W_J and the $\mathcal{H}_J \otimes \mathcal{H}_J$-module $\mathcal{M}_J$, with basis (m_v^J), where v runs through the $B_J \times B_J$-orbits in $Y = \overline{X_J}$, i.e. the $B \times B$-orbits in Y. There is an injective module homomorphism

$$\alpha : \mathcal{M}_J \to \mathcal{M},$$

with $\alpha(m^J_{I,a,b}) = m_{I,w_{0,D}w_{0,J}a,b}$.

Denote by Δ_J the duality map of $\mathcal{M}_J$. The equality (12) will follow from

$$\alpha \circ \Delta_J = \Delta \circ \alpha. \tag{13}$$

To prove (13) notice that under the fibration map $X \to G/P_J^- \times G/P_J$, the orbit $[J, w_{0,d}w_{0,J}, 1]$ is mapped onto a point. The fiber over that point is isomorphic to Y, whence a $B \times B$-equivariant closed embedding $i : Y \to X$, with $i(B_J) = [J, w_{0,d}w_{0,J}, 1]$.

Denote by Δ_X and Δ_Y Verdier duality in the derived category of l-adic sheaves on X, respectively Y. We then have

$$i_* \circ \Delta_Y = \Delta_X \circ i_*,$$

because i is a proper morphism. Eq. (13) is a consequence of this equality, observing that α comes from i_*. □

Lemma 3.7 provides the finishing touch to the proof of part (i) of Proposition 3.5. Part (ii) follows from [KL, 2.1(i)]. Lemma 3.7 is a particular case of the formula of the following lemma (which was pointed out to me by W. van der Kallen). Notations are as before.

3.8. Lemma. *Let $x \in W^J$, $I \subseteq J$, $b \in W_J$. Then*

$$b_{[I,a,b],[J,x,1]} = (-1)^{l(a)+l(b)+l(x)}\bigl(1-u^2\bigr)^{|J-I|} R_{xb,a}\bigl(u^2\bigr). \tag{14}$$

Proof. Recall that by Lemma 3.4(ii) and Corollary 2.5(i) the left-hand side is $\neq 0$ if and only if $I \subseteq J$, $b \in W_J$, and $xb \leqslant a$. We prove (14) by descending induction on $l(x)$. If x is the maximal element of W^J the formula holds by the previous lemma.

Assume (14) holds for x and that $s \in S$, $sx < x$. Then $sx \in W^J$. From (9) we see that

$$b_{[I,a,b],[J,sx,1]} = \begin{cases} b_{(s,1).[I,a,b],[J,x,1]} & \text{if } d((s,1).[I,a,b]) < d([I,a,b]), \\ (1-u^2)b_{[I,a,b],[J,x,1]} + u^2 b_{(s,1).[I,a,b],[J,x,1]} & \\ & \text{if } d((s,1).[I,a,b]) > d([I,a,b]). \end{cases}$$

For the action of $(s, 1)$ see Section 3.3. We put

$$b_{[I,a,b],[J,x,1]} = (-1)^{l(a)+l(b)+l(x)}(1-u^2)^{-1}c_{x,a,b}.$$

We then have to prove that $c_{x,a,b} = R_{xb,a}$. In applying the formulas there are four cases to be dealt with.

(1) $sa > a$ and $sa \in W^I$. Then $c_{sx,a,b} = c_{x,sa,b} = R_{xb,sa}$ by induction. Since $sxb < xb$, $sa > a$ this equals $R_{sxb,a}$ by the first formula (11).

(2) $sa = at$ with $t \in W_I$ and $bt < b$. Now $c_{sx,a,b} = c_{x,a,bt} = R_{xbt,a}$. By the first formula (11) for right action of t and left action of s,

$$R_{xbt,a} = R_{xb,at} = R_{xb,sa} = R_{sxb,a}.$$

(3) $sa < a$. In this case

$$c_{sx,a,b} = (u^2-1)c_{x,a,b} + u^2 c_{x,sa,b} = (u^2-1)R_{xb,a} + u^2 R_{xb,sa}.$$

This equals $R_{sxb,a}$ by the formulas (11).

(4) $sa = at$ with $t \in W_I$ and $bt > b$. Now

$$\begin{aligned} c_{sx,a,b} &= (u^2-1)c_{x,a,b} + u^2 c_{x,a,bt} = (u^2-1)R_{xb,a} + u^2 R_{xbt,a} \\ &= R_{xb,at} \end{aligned}$$

by the second formula (11) for right action of t. Moreover,

$$R_{xb,at} = R_{xb,sa} = R_{sxb,a}$$

by the first formula (11). □

3.9. Remark. In fact, (14) holds for every $x \in W^I$.

We claim that if $x \in W^I$ and $s \in S$ are such that $sx < x$, validity of (14) for sx implies validity for x. By the lemma, validity of (14) for x implies validity for $x = 1$. The claim will then imply validity for any $x \in W^I$.

To prove the claim we use the notations of the proof of Lemma 3.8. Assume that (14) holds for sx. We consider the four cases of the proof of Lemma 3.8. In case (1), $c_{sx,a,b} = c_{x,sa,b}$. Assuming that $c_{sx,a,b} = R_{sxb,a}$, we have $c_{x,sa,b} = R_{xb,sa}$, whence $c_{x,a,b} = R_{xb,a}$ for $sa < a$.

In case (2) a similar argument shows that $c_{x,a,bt} = R_{xbt,a}$, whence $c_{x,a,b} = R_{xb,a}$ if $bt < b$.

In case (3) we find, using the result of case (1),

$$c_{sx,a,b} = R_{sxb,a} = (u^2-1)c_{x,a,b} + u^2 c_{x,sa,b} = (u^2-1)R_{xb.a} + u^2 c_{x,sa,b}.$$

From (11) we see that $c_{x,sa,b} = R_{xb,sa}$, whence $c_{x,a,b} R_{xb,a}$ if $sa > a$. Case (4) is dealt with in a similar manner.

We conclude this section with some additional results about the polynomials $b_{w,v}$ $(v, w \in V)$.

3.10. Lemma.
(i) $b_{w,v}(1) = \delta_{w,v}$.
(ii) $b'_{w,v}(1) = -1$ *if there is a reflection* $r \in \boldsymbol{W}$ (*not necessarily simple*) *with* $w = r.v \leqslant v$.
(iii) *Let* $v = [J, c, d]$, $w = [I, a, b]$. *Then* $b'_{w,v}(1) = -1$ *if* $I \subset J$, $|J - I| = 1$, *and there exists* $f \in W_J$ *with* $a = cf$, $b = df$.
(iv) *In the cases not covered by* (ii) *and* (iii), *we have* $b'_{w,v}(1) = 0$.

Proof. It follows from (9) that if $t \in \boldsymbol{S}$ we have $b_{w,v}(1) = b_{t.w,t.v}(1)$. Let $v = [J, c, d]$, $I = [I, a, b]$. Using the preceding formula the proof of (i) is reduced to the case $c = d = 1$. In that case (14) shows that $b_{w,v} = 0$ unless $I = J$ and $a = b = 1$. (i) follows.

We prove the other assertions by induction on $d(v)$. It is a bit easier to work with the $\tilde{b}_{w,v}$ (introduced in the proof of Proposition 3.5). Using (i) one sees that $\tilde{b}'_{w,v}(1) = 2b'_{w,v}(1)$.

Let $s \in S$, $t = (1, s)$. It follows from (10) that if $d(t.v) < d(v)$

$$\tilde{b}'_{w,v}(1) = \begin{cases} \tilde{b}'_{t.w,t.v}(1) & \text{if } d(t.w) < d(w), \\ -2\tilde{b}_{w,t.v}(1) + \tilde{b}'_{t.w,t.v}(1) & \text{if } d(t.w) > d(w). \end{cases}$$

If $w = t.v < v$ we have $b_{t.w,t.v} = 0$ by Lemma 3.4(ii), and then (i) shows that $\tilde{b}'_{w,v} = -2$, proving (ii). If $w \neq t.v$ the preceding formulas imply that

$$b'_{w,v}(1) = b'_{t.w,t.v}(1).$$

By induction, the proof of (ii)–(iv) is then reduced to the case $d = 1$. In that case we have the explicit formula (14) for $b_{w,v}$. It implies that for $I = J$

$$b'_{w,v}(1) = R'_{cb,a}(1).$$

It is known (see [GJ, 2.2]) that for $u, z \in W$ we have $R'_{u,z}(1) = 0$, except when there is a reflection $r \in W$ with $u = r.z < z$, in which case $R'_{u,z}(1) = 1$. This implies that for $I = J$, $d = 1$, $\tilde{b}_{w,v} = 0$ unless there is a reflection $r \in W$ with $rc = ba^{-1}$, which means that $(r, 1).v = w$, proving (ii) and (iv) if $I = J$.

In the case $I \subset J$, $I \neq J$, $d = 1$, it is clear from (14) that $b'_{w,v}(1) = 0$ if $|J - I| > 1$. If $|J - I| = 1$, (14) gives that $b'_{w,v}(1) = 0$ unless $a = cb$, in which case it equals -1, in accordance with (ii) and (iv). This concludes the proof of Lemma 3.10. □

4. Intersection cohomology of $B \times B$-orbit closures and Kazhdan–Lusztig polynomials

4.1. The notations are as in the preceding section. For $v \in V$ let $\mathcal{I} = \mathcal{I}_v$ be the intersection cohomology complex of the closure $\bar{v}$, i.e. the irreducible perverse sheaf on X supported by $\bar{v}$ whose restriction to v is $E[d(v)]$. It defines an element of $\mathcal{M}$:

$$c_v = u^{-d(v)} \sum_{w \in V} c_{w,v} m_w \quad \text{with} \quad c_{w,v} = \sum_i (-1)^i \left(\sum_{\rho_i} m_{i,\rho_i} \rho_i \right),$$

where the ρ_i run through the images in C of the eigenvalues of the Frobenius map of the stalk $\mathcal{H}^i(\mathcal{I}_v)_a$ in $a \in x(\boldsymbol{F}_q)$, the m_{i,ρ_i} denoting multiplicities (see [MS, 3.1.2]). We have $c_{v,v} = 1$, and $c_{w,v} = 0$ if $w \not\leqslant v$.

4.2. Theorem.
(i) *The $c_{w,v}$ are polynomials in u^2 with positive integral coefficients*;
(ii) *$\mathcal{I}$ is even, i.e. $\mathcal{H}^i(\mathcal{I}) = 0$ if $i + d(v)$ is odd.*

Proof. Again we use the result from [B2, 3.1, Theorem] that in a point of an orbit v there is a contractible transversal slice. By [MS, 2.3.3] it then follows that $\mathcal{I}$ is punctually pure, i.e. that all eigenvalues of the Frobenius map of a stalk $\mathcal{H}^i(\mathcal{I}_v)_a$ ($a \in \bar{v}(\boldsymbol{F}_q)$) have absolute values $q^{(i+d(v))/2}$. The fact that the $b_{x,v}$ are polynomials, proved in Lemma 3.4, implies (i) and (ii). We refer to [MS, 3.4.3 and proof of 7.1.2(ii)]. □

Now assume that we work over an arbitrary algebraically closed field.

4.3. Corollary. *$\mathcal{I}$ is even.*

Proof. Since this is true when k is the algebraic closure of a finite field by the theorem, it is true for any k by a familiar reduction procedure (see [BBD, Section 6]). It also follows that the result holds over $\mathbb{C}$, relative to the classical topology. □

We can now discard the big ring $\mathcal{R}$. View the $c_{w,v}$ as polynomials in u^2. They are the "Kazhdan–Lusztig polynomials" for $\mathcal{M}$, characterized by properties of the usual kind.

4.4. Proposition. *The $(c_{w,v})_{x,v \in V}$ are the uniquely determined polynomials with the following properties*:

(a) $c_{v,v} = 1$ *and* $c_{w,v} = 0$ *if* $w \not\leqslant v$;
(b) *if* $w < v$ *the u-degree of* $c_{w,v}(u^2)$ *is* $< d(v) - d(w)$;

(c) $u^{-d(v)}\sum_w c_{w,v}(u^2)m_w$ *is invariant under* Δ.

Proof. Our polynomials $c_{w,v}$ have these properties. For (a) this is clear, and (b), (c) reflect properties of the perverse sheaf $\mathcal{I}$, viz. the support conditions and duality.

The uniqueness follows from the following identity (cf. [KL, 2.2]):

$$u^{-d(v)+d(w)}c_{w,v}\bigl(u^2\bigr) - u^{d(v)-d(w)}c_{w,v}\bigl(u^{-2}\bigr) = \sum_{w<y\leqslant v} u^{d(v)+d(w)-2d(y)}c_{y,v}\bigl(u^{-2}\bigr)b_{w,y}\bigl(u^2\bigr).$$

This can be written in a somewhat less cumbersome form. Write

$$\tilde{c}_{w,v}(u) = u^{d(v)-d(w)}c_{w,v}\bigl(u^{-2}\bigr), \qquad \tilde{b}_{w,v}(u) = u^{-d(v)+d(w)}b_{w,v}\bigl(u^2\bigr).$$

For $w \neq v$, $\tilde{c}_{w,v}$ is a polynomial in u without constant term, an integral linear combination (with coefficients $\geqslant 0$) of powers u^j with $j + d(v) + d(w)$ even. Moreover, $\tilde{b}_{w,v}$ is a Laurent polynomial in u, a linear combination of powers of u satisfying the previous parity condition. The preceding formula can be rewritten as

$$\tilde{c}_{w,v}\bigl(u^{-1}\bigr) - \tilde{c}_{w,v}(u) = \sum_{w<y\leqslant v} \tilde{c}_{y,v}(u)\tilde{b}_{w,y}(u). \tag{15}$$

It follows from Proposition 3.5(ii) that the Laurent polynomial $\tilde{b}_{w,v}(u)$ is of the form $f(u-u^{-1})$, where $f \in \mathbb{Z}[T]$. $\square$

4.5. The formula (15) leads to an inductive procedure to determine the Kazhdan–Lusztig polynomials $c_{w,v}$, via the Laurent polynomials $\tilde{c}_{w,v}$. Namely, the right-hand side of (15) is a Laurent polynomial without constant term and $-\tilde{c}_{w,v}$ is its polynomial part. One needs to know the Laurent polynomials $\tilde{b}_{w,y}$. They can be determined inductively, as in the proof of Proposition 3.5: reduce to the case that $v = [J, x, 1]$ by using (10) and then apply Lemma 3.8.

The procedure has been implemented by W. van der Kallen in a Mathematica program. Among other things, he computed the $c_{w,v}$ in the case that G is simple of rank two and that $v = [D, 1, 1]$, i.e. the Borel subgroup B of G. See Appendix A for more details.

The computations give that in type A_2 (i.e. for $G = PGL_3$) we have $c_{[I,x,y],B} = 1$ unless $I = \emptyset$ and either $x = st$, $y = t, 1$ or $x = sts$, $y = s, t, 1$ (where s and $t \neq s$ are simple reflections). In these cases $c_{[\emptyset,x,y],B} = 1 + u^2$, except when $x = sts$, $y = 1$, in which case this polynomial is $(1+u^2)^2$.

In general, $P_{y,x} = c_{[\emptyset,x,y],B}$ $(y \leqslant x)$ is a polynomial in u^2 whose u-degree is $< 2(r + l(x) - l(y))$, where r is the rank of G (by Proposition 4.4(b) and (5)). These polynomials remind one of the usual Kazhdan–Lusztig polynomials, introduced in [KL]. One can wonder whether $P_{y,x}$ also have some bearing on representation theory. It would be interesting to have a more direct combinatorial definition of these polynomials.

We next give some other consequences of (15).

4.6. Proposition.
(i) *If* $w \leqslant v$ *then* $c_{w,v}$ *has constant term* 1.
(ii) $c_{w,v} \neq 0$ *if and only if* $w \leqslant v$.
(iii) *If* $w < v$ *then* $\sum_{w \leqslant y \leqslant v} (-1)^{d(y)-d(w)} = 0$.

Proof. (i) is equivalent to the statement that the polynomial (in u^{-1}) $\tilde{c}_{w,v}(u^{-1})$ has leading term $(u^{-1})^{d(v)-d(w)}$. Now $\tilde{c}_{y,v}$ is a polynomial in u without constant term if $y < v$ and $\tilde{b}_{w,y}$ is a Laurent polynomial in u with lowest term $(u^{-1})^{d(y)-d(w)}$. It follows that for $w < y \leqslant v$, $\tilde{c}_{y,v}\tilde{b}_{w,y}$ is a Laurent polynomial in u whose lowest term is $(u^{-1})^m$ with $m \leqslant (d(y) - d(w))$, equality occurring only if $y = v$. This implies that the left-hand side of (15) has lowest term $(u^{-1})^{d(v)-d(w)}$. This lowest term must occur in $\tilde{c}_{w,v}$ and (i) follows.

(ii) is a consequence of (i) and Proposition 4.4(a).

To prove (iii), consider the leading coefficient in u in both sides of (15). By (i) this is -1 in the left-hand side. In the right-hand side the leading coefficient is $\sum_{w<y\leqslant v} (-1)^{d(v)-d(y)}$, as follows from Proposition 3.5. The formula of (iii) follows. □

Remark. (iii) implies that the Möbius function of the ordered set V is given by $\mu(w,v) = (-1)^{d(v)-d(w)}$ $(w \leqslant v)$. This is similar to a result of Verma for the Bruhat order of W. In [KL, 3.3(b)] Verma's result is deduced from properties of Kazhdan–Lusztig polynomials. It can also be proved along the lines of the proof of Proposition 4.6(iii).

If $w = [I, a, b]$ and $I \subset J$ we put $\pi_{I,J}(w) = [J, c, d]$, where $c \in W^J$ is such that $a \in cW_J$ and $d = bc^{-1}a$. We denote by $R(\boldsymbol{W})$ the set of reflections in $\boldsymbol{W}$.

4.7. Proposition. *Let* $v \in V$ *and assume that* $\bar{v}$ *is rationally smooth. Then for any* $w < v$

$$\bigl|\{r \in R(\boldsymbol{W}) \mid w < r.w \leqslant v\}\bigr| + \bigl|\bigl\{J \subset D \bigm| J \subseteq I,\ |J - I| = 1,\ \pi_{I,J}(w) \leqslant v\bigr\}\bigr|$$

equals $d(v) - d(w)$.

Proof. That $\bar{v}$ is rationally smooth means that the intersection cohomology complex $\mathcal{I}_v$ is $E_{\bar{v}}[\dim v]$ or, equivalently, that $c_{w,v} = 1$ for $w \leqslant v$. Then $\tilde{c}_{w,v}(u) = u^{d(v)-d(w)}$ for $w \leqslant v$. Inserting this into (15) and taking derivatives of both sides for $u = 1$, one obtains for $w < v$

$$-2\bigl(d(v) - d(w)\bigr) = \sum_{w<y\leqslant v} \bigl(\bigl(d(v) - d(y)\bigr)b_{w,y}(1) + \tilde{b}'_{w,y}(1)\bigr).$$

The asserted equality is now a straightforward consequence of Lemma 3.10. □

Remark. It was pointed out by M. Brion that the right-hand side of the formula of Proposition 4.7 can be interpreted as the number of $(T \times T)_w$-stable curves in the slice to $\bar{v}$ along w, see [B2, 1.4, Corollary 2].

An application of Proposition 4.7 is the following result due to Brion, proved in a somewhat different manner in [B2, 3.3].

4.8. Corollary. *Assume that G is simple. If $w \in W$ the large Schubert variety $S_w = \overline{BwB}$ is rationally smooth if and only if $G \simeq PGL_2$ or $w = w_{0,D}$.*

Proof. Take in the proposition $v = BwB = [D, 1, w]$ and $w = [\emptyset, w_{0,D}, 1]$. By Corollary 2.5(iii), $w \leqslant [D, 1, 1] \leqslant v$ (w is the unique $B \times B$-fixed point in S_w). We have $d(v) = d + r + l(w)$, where $d = l(w_{0,D})$ and r is the rank of G. If s is any reflection in W then $(s, 1).w = [\emptyset, sw_{0.D}, 1]$ and $(1, s).w = [\emptyset, w_{0.D}, s]$. From Corollary 2.5(iii) we then see that all $r \in R(\boldsymbol{W})$ satisfy $w < r.w \leqslant [D, 1, 1] \leqslant v$. Further, if $\alpha \in D$ and $s \in S$ is the corresponding simple reflection then $\pi_{\emptyset,\{\alpha\}}(w) = [\{\alpha\}, w_{O,D}s, s]$. By Corollary 2.5(iii) we have $\pi_{\emptyset,\{\alpha\}}(w) \leqslant v$ if $s \leqslant w_{0,D}s$. If this is not the case and $r > 1$, we must have $w_{0,D}s \in W_{D-\{\alpha\}}$. Since $l(w_{0,D}s) = d - 1$, $w_{0,D}s$ has to be the longest element of $W_{D-\{\alpha\}}$. This implies that $s \in W^{D-\{\alpha\}}$ whence $s.\beta \in R^+$ for all simple roots $\beta \neq \alpha$, which is impossible if $r > 1$ and R is irreducible (G being simple).

It follows that if $r > 1$ the number given by the displayed formula in the Proposition is at least $2d + r$. Since $d(v) - d(w) = d + r + l(w)$, Proposition 4.7 gives that if $r > 1$ rational smoothness of S_w implies

$$r + d + l(w) \geqslant 2d + r,$$

which can only be if $w = w_{0,D}$.

Conversely, if $w = w_{0,D}$ then $S_w = X$ is smooth. To finish the proof of Corollary 4.8 it remains to be shown that if $G = PGL_2$, $\overline{B}$ is rationally smooth. The wonderful compactification X of PGL_2 is isomorphic to projective space $\boldsymbol{P}^3$, viewed as the set of lines in the space of 2×2-matrices. It follows that $\overline{B}$ is isomorphic to $\boldsymbol{P}^2$, hence is smooth. (Rational smoothness of $\overline{B}$ in this case can also be proved by hand, using (15).) □

4.9. Global intersection cohomology. We next establish parity of global intersection cohomology of orbit closures. If X is any irreducible variety, define its global intersection cohomology groups by

$$IH^i(X) = \mathcal{H}^i\big(X, \mathcal{I}_X[-\dim X]\big),$$

the hypercohomology of a shift of the intersection cohomology complex $\mathcal{I}_X$ of X. The shift is added in order to recover ordinary cohomology if X is smooth.

As before, we work over the algebraic closure of a sufficiently large field $\boldsymbol{F}_q$, over which X is defined. Let X_0 be the underlying $\boldsymbol{F}_q$-variety. Then $\mathcal{I}_X$ comes from a perverse sheaf $\mathcal{I}_0$ on X_0. Moreover, $\mathcal{I}_0$ is pure of weight 0 (see [BBD, 5.3.2]). We have a Frobenius endomorphism F of the intersection cohomology groups. If X is projective and $IH^i(X) \neq 0$, all absolute values of the eigenvalues of F on a $IH^i(X)$ are $q^{i/2}$, as follows from [BBD, Section 5].

Now let $X = \bar{v}$, where $v \in V$.

4.10. Lemma. *The eigenvalues of F on a nonzero intersection cohomology group are integral powers of q.*

Proof. Let A be the shifted intersection cohomology complex $\mathcal{I}_X[-\dim X]$. Put

$$X_n = \bigcup_{\substack{w \in V,\ w \leqslant v, \\ \dim w \leqslant n}} w,$$

which is a closed subset of X, coming from an $\boldsymbol{F}_q$-subvariety $(X_n)_0$ of X_0. Clearly, $X_{\dim v} = X$. We show by induction on n that for each n the eigenvalues of F on a nonzero hypercohomology group $\mathcal{H}^i(X_n, A)$ are integral powers of q. We have exact sequences of hypercohomology groups

$$\cdots \to \mathcal{H}^{i-1}(X_{n-1}, A) \to \mathcal{H}^i_c(X_n - X_{n-1}, A) \to \mathcal{H}^i(X_n, A) \\ \to \mathcal{H}^i(X_{n-1}, A) \to \cdots,$$

the arrows commuting with the respective Frobenius endomorphisms. Moreover,

$$X_n - X_{n-1} = \coprod_{\dim w = n} w.$$

A straightforward argument now shows that it suffices to prove that the eigenvalues of F on a nonzero group $\mathcal{H}^i_c(w, A)$ are integral powers of q.

We have a spectral sequence

$$H^i_c\big(w, H^j(A)\big) \Rightarrow \mathcal{H}^{i+j}_c(w, A),$$

from which we conclude that it suffices to prove a similar assertion for the groups $H^i_c(w, H^j(A))$. The restriction of the locally constant sheaf $H^j(A)$ to w is $B \times B$-equivariant. Since the isotropy groups in $B \times B$ of the points of w are connected (see Lemma 1.3(iv)) this restriction is constant. By Theorem 4.2(ii) $H^j(A) = 0$ if j is odd and it follows from Theorem 4.2(i) (cf. the description of $c_{w,v}$ given in Section 4.1) that all eigenvalues of F on the stalk $H^j(A)_x (x \in w(\boldsymbol{F}_q))$ are q^j. This reduces the proof to showing that the eigenvalues of F on $H^i_c(w, E)$ are powers of q. This follows from the fact that w is isomorphic over $\boldsymbol{F}_q$ to the product of a torus and an affine space. □

4.11. Theorem. $IH^i(X) = 0$ *if i is odd.*

Proof. All absolute values of an eigenvalue of F on a nonzero group $IH^i(X)$ are $q^{i/2}$. By Lemma 4.10 this must be an integral power of q, which can only be if i is even.

Again, the parity result is true in any characteristic and over $\mathbb{C}$, in the classical context. □

The arguments of the proof can be extended a bit, so as to give a description of the intersection cohomology Poincaré polynomial of $X = \bar{v}$.

4.12. Corollary. $\sum_i \dim IH^i(X)\, t^i = \sum_{w \leqslant v} t^{2\dim w}\bigl(1 - t^{-2}\bigr)^{|I_w|} c_{w,v}\bigl(t^2\bigr).$

Proof. It follows from the theorem that all eigenvalues of F on $IH^{2i}(X)$ are q^i. A being as in the proof of Lemma 4.10, it also follows that

$$\sum_{i \geqslant 0} \dim IH^{2i}(X) q^i = \sum_i (-1)^i \operatorname{Tr}\bigl(F, \mathcal{H}^i(X, A)\bigr).$$

By a result of Grothendieck, the right-hand side equals

$$\sum_{x \in \bar{v}(\boldsymbol{F}_q),\, i} (-1)^i \operatorname{Tr}\bigl(F, H^i(A)_x\bigr).$$

Now $\bar{v}(\boldsymbol{F}_q) = \coprod_{w \leqslant v} w(\boldsymbol{F}_q)$, and the number of points of $w(\boldsymbol{F}_q)$ equals $q^{\dim w}(1 - q^{-1})^{|I_w|}$. Moreover, for all $x \in w(\boldsymbol{F}_q)$

$$\sum_i (-1)^i \operatorname{Tr}\bigl(F, H^i(A)_x\bigr) = c_{w,v}(q).$$

We conclude that the difference of the two sides of the asserted formula vanishes for $t = q^{1/2}$. But it then also vanishes for all powers $(q^{1/2})^n$, hence the difference is identically zero. □

Example. Let $G = PGL_3$, $v = B$. The Kazhdan–Lusztig polynomials were described in Section 4.5. Cut the sum of Corollary 4.12 into pieces corresponding to the cells of Proposition 2.10. A straightforward computation gives for the intersection cohomology Poincaré polynomial of $\overline{B}$,

$$1 + 4t^2 + 9t^4 + 9t^6 + 4t^8 + t^{10}.$$

For some other cases the polynomials are given in Appendix A.

It was pointed out by M. Brion that the description of global intersection cohomology can also be obtained as an application of the results of [BJ].

5. Nonconstant local systems

In this section we discuss the generalization of Theorem 3.2 to the case of intersection cohomology complexes for not necessarily constant local systems on the $B \times B$-orbits v in X. These are the local systems having a weight for the $B \times B$-action (see [MS, 2.2]).

If $v = [I, x, v]$ (as before), we write $I = I_v$.

5.1. Assume that k is the algebraic closure of a finite field $\boldsymbol{F}_q$, over which all objects which occur are defined. Let $\mathcal{R}$ be as in Section 3. Following [MS] we introduce the free R-module $\mathcal{N}$ with basis $m_{\xi,v}$ where $v \in V$ and $\xi \in \widehat{X}(T \times T/(T \times T)_v) = \widehat{X}(T_{I_v}) = \widehat{X}_{I_v} \subset \widehat{X}$ (see Section 1.7). Then $m_{\xi,v}$ is the class in the Grothendieck group $\mathcal{K}(\mathcal{A}_X)$ (see Section 3.1) of $(S_{\xi,v}, \phi)$, where $S_{\xi,v}$ is the sheaf which restricts to ξ on v and to 0 on the complement of v, ϕ being a Frobenius map. The module $\mathcal{M}$ of Section 2 is the submodule with basis $m_v = m_{0,v}$ $(v \in V)$.

Let $\mathcal{K}$ be the algebra over $\mathbb{Z}[u, u^{-1}]$ with basis $(e_{\xi,w})$, where $w \in W$ and $\xi \in \widehat{X} = \widehat{X}(T)$, the multiplication being defined by the following rules (see [MS, 4.3.2 and 3.2.3]):

$$e_{\xi,x} e_{\eta,y} = 0 \quad \text{for } \xi \neq y.\eta,$$

and for $s \in S$, $\xi = y.\eta$:

$$e_{\xi,s} e_{\eta,y} = \begin{cases} e_{\eta,sy} & \text{if } sy > y, \\ (u^2 - 1)e_{\eta,y} + u^2 e_{\eta,sy} & \text{if } sy < y \text{ and } \langle y\eta, \check{\alpha}\rangle = 0, \\ u^2 e_{\eta,sy} & \text{if } sy < y \text{ and } \langle y\eta, \check{\alpha}\rangle \neq 0. \end{cases}$$

$\check{\alpha}$ is the element $\alpha^\vee \otimes 1$ of $\widehat{Y}(T) = Y(T) \otimes (\mathbb{Z}_{(p)}/\mathbb{Z})$, where $\alpha^\vee$ is the cocharacter defined by α.

Moreover (see [MS, 3.2]),

$$e_{\xi,1} e_{\eta,y} = \delta_{\xi,y.\eta} e_{\eta,y}.$$

If $\xi \in \widehat{X}$ we denote by R_ξ the closed subsystem of R consisting of the roots α with $\langle \xi, \check{\alpha}\rangle = 0$. Its Weyl group is W_ξ. It is a normal subgroup of the isotropy group W'_ξ of ξ in W.

By [MS, 3.2] we have a structure of $\mathcal{K} \otimes_{\mathbb{Z}[u,u^{-1}]} \mathcal{K}$-module on $\mathcal{N}$. The next lemma, which generalizes Lemma 3.2, describes the module structure.

Let $v = [I, x, w] \in V$. For $\xi \in \widehat{X}_I$ we put $m_{\xi,I,x,w} = m_{\xi,v}$. In (ii) we use the notations introduced before Lemma 3.2.

5.2. Lemma. *Let $x \in W^I$, $w \in W$, $\xi \in \widehat{X}_I$, $\eta \in \widehat{X}$, and $s \in S$.*

(i) $e_{\eta,(s,1)}.m_{\xi,I,x,w} = 0$ *if* $\eta \neq x^{-1}.\xi$ *and* $e_{\eta,1} m_{\xi,I,x,w} = \delta_{\eta,x.\xi} m_{\xi,I,x,w}$.

(ii) *If* $\eta = x^{-1}.\xi$, *the product of* (i) *equals*
   (a) $(u^2-1)m_{\xi,I,x,w} + u^2 m_{\xi,I,sx,w}$ *in case* (A) *if* $x.\alpha \in R_\xi$, *and* $u^2 m_{\xi,I,sx,w}$ *otherwise*;
   (b1) $m_{\xi,I,x,w\sigma}$ *in case* (B) *if* $w\sigma > w$;
   (b2) $(u^2-1)m_{\xi,I,x,w} + u^2 m_{\xi,I,x,w\sigma}$ *in case* (B) *if* $w\sigma < w$, $w.\alpha \in R_\xi$, *and* $u^2 m_{\xi,I,x,w\sigma}$ *if* $w\sigma < w$, $w.\alpha \notin R_\xi$;
   (c) $m_{\xi,I,sx,w}$ *in case* (C).

(iii) $e_{\eta,(1,s)}.m_{\xi,I,x,w} = 0$ *if* $\eta \neq -w^{-1}.\xi$.

(iv) *If* $\eta = -w^{-1}.\xi$, *the product of* (iii) *equals* $m_{\xi,I,x,sw}$ *if* $sw > w$. *If* $sw < w$ *it equals* $(u^2-1)m_{\xi,I,x,w} + u^2 m_{\xi,I,x,sw}$ *if* $w.\alpha \in R_\xi$, *and* $u^2 m_{\xi,I,x,sw}$ *otherwise.*

Proof. The first point of (i) follows from [MS, 3.2.3] and the second point is an easy consequence of the definitions of [MS, 3.2.1].

The formulas of [MS, 4.3.1] for the cases IIa and IIb (proved in [MS, 4.3.4, 4.3.9]) give formulas like those of (ii), except that at first sight on the right-hand side other elements of $\widehat{X}_I$ might appear. Consider, for example, case (a) with $x.\alpha \in R_\xi$. Then [MS] shows, using Lemma 1.8, that there is ξ' such that

$$e_{x^{-1}.\xi,s}.m_{\xi,I,x,w} = m_{\xi',I,sx,w}.$$

It follows from the definition of ϕ_v (see Lemma 1.8) that $\phi_{(s,1).v} = \phi_v \circ (s,1)$, whence $\widehat{\phi_{(s,1).v}} = (s,1) \circ \widehat{\phi_v}$. By the results of [MS] $\widehat{\phi_{(s,1).v}}(\xi') = (s,1).\widehat{\phi_v}(\xi)$. By Lemma 1.8

$$y^{-1}.\xi' = \widehat{\rho_{(s.1).v}}(\xi') = \widehat{\rho_v}(\xi) = y^{-1}.\xi,$$

whence $\xi' = \xi$. In the other cases the proof that only ξ will occur is similar. This will prove (ii). The proofs of (iii) and (iv) are similar to the proof of (i), respectively (ii). $\square$

Lemma 5.2 shows that the $m_{\xi,v}$ with a fixed $\xi \in \widehat{X}$ span a submodule $\mathcal{M}_\xi$ of $\mathcal{N}$ which is stable under the action of $\mathcal{H} \otimes \mathcal{H}$. Clearly, $\mathcal{N}$ is the direct sum of the $\mathcal{M}_\xi$.

5.3. As for the module $\mathcal{M}$, there exists a semilinear involutorial endomorphism Δ of $\mathcal{N}$, coming from Verdier duality, see [MS, 3.3]. It will follow from Lemma 5.4(ii) that Δ maps $\mathcal{M}_\xi$ onto $\mathcal{M}_{-\xi}$.

For $s \in S$, $\xi \in \widehat{X}$, $m \in \mathcal{N}$ we have

$$\Delta(e_{\xi,(s,1)}.m) = \begin{cases} u^{-2}\big(e_{-\xi,(s,1)} + \big(1-u^2\big)e_{-\xi,1}\big).\Delta(m) & \text{if } s \in W_\xi, \\ u^{-2}e_{-\xi.(s,1)}.\Delta(m) & \text{if } s \notin W_\xi, \end{cases} \tag{16}$$

and similarly for $e_{\xi,(1,s)}$. See [MS, 5.1].

5.4. Lemma.

(i) *There exist elements* $b_{\eta,w;\xi,v} \in \mathcal{R}$ *such that*

$$\Delta(m_{\xi,v}) = u^{-2d(v)} \sum_{w\in V,\ \eta\in\widehat{X}_{I_w}} b_{\eta,w;\xi,v} m_{\eta,w}.$$

(ii) *If* $b_{\eta,w;\xi,v} \neq 0$ *then* $w \leqslant v$ *and* $\widehat{\phi_w}.\eta = -\widehat{\phi_v}.\xi$. *Moreover,* $b_{-\xi,v;\xi,v} = 1$.

Proof. The proof of (i) is like the proof of Lemma 3.4(i). The first point of (ii) follows from the fact that the Verdier dual $\Delta(S_{\xi,v})$ is zero outside $\bar{v}$. The restriction of $\Delta(S_{\xi,v})$ to v is $-\xi$ shifted by $2d(v)$, which implies the last statement of (ii). The second one follows from [MS, 3.4]. $\square$

5.5. Proposition.

(i) *The* $b_{\eta,w;\xi,v}$ *are polynomials in* $\mathbb{Z}[u^2]$.
(ii) *If* $b_{\eta,w;\xi,v} \neq 0$ *then* $\xi = -\eta$. *In particular* $\xi \in \widehat{X}_{I_w}$.

Proof. The proof of (i) is along the lines of the proof of Proposition 3.5(i). We shall not spell out the details. Let $v = [I, x, w] \in V$ and assume that there is $t \in \boldsymbol{S}$ with $d(t.v) < d(v)$. Using Lemma 5.2 and (16) one reduces the proof to the case that $v = [J, w_{0,D}w_{0,J}, 1]$. (The important fact for the proof of (ii) is that by the formulas of Lemma 5.2, $e_{\eta,t}.m_{\xi,v}$ lies in $\mathcal{M}_\xi$.)

Assume that v has this form, and assume also that $J = D$. So $v = B$. We then have to compute the $b_{\eta,[I,1,1];\xi,B}$. First note that by Lemmas 5.4(ii) and 1.8, $b_{\eta,[I,1,1];\xi,[D,1,1]} = 0$ if $\eta \neq -\xi$.

Assume that $\eta = -\xi$. Then $\xi \in \widehat{X}_I$. Proceeding now as at the end of the proof of Lemma 3.6, we see that we have to compute the cohomology of the torus T_{D-I} with values in the restriction to T_{D-I} of the local system ξ on T. But the restriction map $\widehat{X} \to \widehat{X}(T_{D-I})$ has kernel $\widehat{X}_I$, so contains ξ. It follows that the restriction of ξ to T_{D-I} is the trivial local system, and the computation is then as in Lemma 3.8, showing that $b_{-\xi,[I,1,1];\xi,B}$ is a nonzero polynomial in u^2 if $\xi \in \widehat{X}_I$.

It remains to deal with the case that $v = [J, w_{0,D}w_{0,J}, 1]$ with J arbitrary. This is done by using an analogue of (13). $\square$

5.6. Intersection cohomology. For $v \in V$ and $\xi \in \widehat{X}_{I_v}$ denote by $\mathcal{I}_{\xi,v}$ the intersection cohomology complex of the closure $\bar{v}$, for the local system ξ on v, i.e., the irreducible perverse sheaf on X which is zero outside $\bar{v}$ and whose restriction to v is $\xi[d(v)]$. As in Section 3.1, it defines an element of $\mathcal{N}$

$$c_{\xi,v} = u^{-d(v)} \sum_{w\in V,\, \eta\in\widehat{X}_{I_w}} c_{\eta,w;\xi,v} m_{\eta,w},$$

the $c_{\eta,w;\xi,v}$ being elements of $\mathcal{R}$.

5.7. Theorem.
(i) *The* $c_{\eta,w;\xi,v}$ *are polynomials in* u^2 *with positive integral coefficients.*
(ii) *If* $c_{\eta,w;\xi,v} \neq 0$ *then* $\xi = -\eta$; *in particular,* $\xi \in \widehat{X}_{I_w}$.
(iii) $\mathcal{I}_{\xi,v}$ *is even.*

Proof. The proof of (i) and (iii) is similar to the proof of the analogous results of Section 4.

(ii) follows from Proposition 5.5(ii), using the inductive description of the $c_{\eta,x;\xi,v}$ contained in [MS, 3.4.2, 3.4.3]. It follows from (iii) that the restriction of $\mathcal{I}_{\xi,v}$ to X_I is zero if $\xi \notin \widehat{X}_I$.

(ii) remains true over any algebraically closed field (cf. Section 3.3). □

6. Arbitrary Coxeter groups

This section deals with a tentative extension to arbitrary Coxeter groups of the constructions of the preceding sections.

6.1. We first give a more intrinsic description of the module $\mathcal{M}$ of Section 2. We now identify D and S. For $I \subset S$ let $\mathcal{M}_I$ be the submodule of $\mathcal{M}$ spanned by the m_v with $I_v = I$. Denote by $\mathcal{H}_I$ the Hecke algebra of the subgroup W_I generated by I. It is a subalgebra of $\mathcal{H}_S = \mathcal{H}$. Denote by j_I the isomorphism of $\mathcal{H}_I$ sending $e_w (w \in w_I)$ to $e_{w_{0,I} w w_{0,I}}$.

Let i be the endomorphism of $\mathcal{M}$ sending $m_{I,x,w}$ to $m_{I,x,w^{-1}}$. Then

$$\big((h,h'),m\big) \mapsto h.m.h' = i\big((h \otimes h').i(m)\big) \quad (h,h' \in \mathcal{H},\ m \in \mathcal{M})$$

defines a structure of $(\mathcal{H},\mathcal{H})$-bimodule on $\mathcal{M}$. In particular, we can view $\mathcal{M}$ as an $(\mathcal{H}_I,\mathcal{H})$-bimodule.

If $\mathcal{N}$ is a (left) $\mathcal{H}_I$-module, denote by ${}_{j_I}\mathcal{N}$ the $\mathcal{H}_I$-module $\mathcal{N}$ twisted by j_I.

6.2. Lemma.
(i) $\mathcal{M} = \bigoplus_{I \subset S} \mathcal{M}_I$.
(ii) *There is an isomorphism* ϕ *of* $(\mathcal{H},\mathcal{H})$*-bimodules of* $\mathcal{M}_I$ *onto the twisted induced module* ${}_{j_S}(\mathcal{H} \otimes_{\mathcal{H}_I} ({}_{j_I}\mathcal{H}))$.
(iii) ϕ *commutes with right* $\mathcal{H}$*-actions.*

Proof. For (iii) notice that the induced module $\mathcal{N}$ of (ii) has a natural right $\mathcal{H}$-action.

(i) is clear. For the proof of (ii) notice that $\mathcal{H}$ is a free right module over $\mathcal{H}_I$ with basis $e_x (x \in W^I)$. It follows that $(e_x \otimes m)_{x \in W^I, m \in \mathcal{M}}$ is a basis of $\mathcal{N}$.

Now define ϕ by

$$\phi(m_{I,x,w}) = e_{\iota_I(x)} \otimes e_w,$$

where $\iota_I(x) = w_{0,D}xw_{0,I}$, as in Lemma 1.3. A straightforward check shows that for $s \in S$

$$\phi(e_{(s,1)}.m_{I,x,w}) = \begin{cases} e_{(s^*,1)}.\phi(m_{I,x,w}) & \text{if } sx \in W^I, \\ e_{\iota_I(x)} \otimes e_{w_{0,I}\sigma w_{0,I}} e_w & \text{if } \sigma = x^{-1}sx \in W_I. \end{cases}$$

This proves (ii).

The proof of (iii) is easy. □

Now assume that (W, S) is an arbitrary Coxeter group (with a finite set of generators S). As in Section 3, we write $\boldsymbol{W} = W \times W$, $\boldsymbol{S} = S \times S$.

For $I \subset S$ let, as before, W_I be the subgroup generated by I and W^I the set of distinguished coset representatives for W_I, i.e. the set of $x \in W$ with $xs > x$ for all $s \in I$.

Let V be the set of triples $[I, x, w]$ with $I \subseteq S$, $x \in W^I$, $w \in W$. We introduce the free $\mathbb{Z}[u, u^{-1}]$-module $\mathcal{M}$ with basis $(m_v)_{v \in V}$. For $v = [I, x, w]$ we write $m_v = m_{I,x,w}$.

As before, $\mathcal{M}$ is a direct sum of submodules $\mathcal{M}_I$ $(I \subset S)$.

6.3. Proposition. *The formulas of Lemma* 3.2 *define a structure of* $\mathcal{H} \otimes \mathcal{H}$*-module on all* $\mathcal{M}_I$.

Proof. It is straightforward to verify that for $t \in \boldsymbol{S}$, $v \in V$

$$e_t^2.m_v = \left(u^2 - 1\right)e_t.m_v + u^2 m_v.$$

To prove Proposition 6.3 it remains to verify that for $t, t' \in \boldsymbol{S}$, the endomorphisms e_t and e_t' of $\mathcal{M}$ verify the appropriate braid relations. This is immediate if one of t, t' is of the form $(1, s)$ with $s \in S$. So assume that $t = (s, 1)$, $t' = (s', 1)$ $(s, s' \in S)$. We may assume that $s \neq s'$. If a braid relation is to be verified, ss' has finite order. Putting $J = \{s, s'\}$ the group W_J is finite.

Fix $v = [I, x, w] \in V$. Let $\mathcal{N}$ be the submodule of $\mathcal{M}$ spanned by the $m_{I,x',w'}$ with $x' \in W_J x$, $w' \in wW_I$. We may assume that x is the unique element of $W^I \cap (W^J)^{-1}$ lying in $W_J x W_I$. One knows that then $W_I \cap x^{-1}W_J x = W_K$, where $K = I \cap x^{-1}.J$ (see [C, p. 65]). Now the w' which occur will lie in a fixed coset modulo W_K. Taking $w \in W^K$ we see that $\mathcal{N}$ can be viewed as a module like $\mathcal{M}_I$, for the Hecke algebra $\mathcal{H}_J$ of the finite Coxeter group W_J. But for such a group we have again the result of Lemma 6.2, which implies that the formulas of Lemma 3.2 define a representation of $\mathcal{H}_J$ in $\mathcal{N}$. This implies that the braid relations hold for the action in $\mathcal{M}$ of $e_{(s,1)}$ and $e_{(s',1)}$. □

6.4. The set V. Suggested by the results of Section 2 we introduce some structure on the set V.

(a) Suggested by (5) we define a dimension function d on V by

$$d\left([I, x, w]\right) = -l(x) + l(w) + |I|.$$

Note that if W is infinite the value of d can be any integer.

(b) We have an action on V of $\boldsymbol{W} = W \times W$, described as in Section 3.3. That we do have a $\boldsymbol{W}$-action follows from Proposition 6.3, specializing u to 1. We also have an action of the monoid $M(W \times W)$ on V.

(c) We next introduce an order relation on V. A simple way of doing this is suggested by Proposition 2.8. Let $\leqslant_1$ and $\leqslant_2$ be defined as in Lemma 2.6. These are orderings. Denote by $\leqslant$ the ordering generated by $\leqslant_1$ and $\leqslant_2$.

6.5. Lemma.

(i) *If* $[I, x', w'] \leqslant [J, x, w]$ *then* $x \leqslant x'$ *and* $-l(x') + l(w') \leqslant -l(x) + l(w)$.

(ii) *Segments in* V *for the ordering* $\leqslant$ *are finite.*

Proof. To prove (i) it suffices that the properties of (i) hold for $\leqslant_1$ and $\leqslant_2$. For $\leqslant_1$ this is immediate. If $[I, x', w'] \leqslant_2 [J, x, w]$ there is $z \in W_J$ with $w' = wz$, $l(w') = l(w) + l(z)$, $xz \leqslant x'$. Moreover, $I \subseteq J$. Then

$$-l(x') + l(w') = -l(x') + l(xz) - l(x) + l(w) \leqslant -l(x) + l(w).$$

Also, $x \leqslant xz \leqslant x'$.

To prove (ii), let

$$[H, x'', w''] \leqslant [I, x', w'] \leqslant [J, x, w].$$

By (i) we have $x' \leqslant x''$ and

$$l(w') \leqslant -l(x) + l(w) + l(x') \leqslant l(w) + l(x'').$$

These inequalities imply that the segments in V for $\leqslant$ are finite, proving (ii).

We prove that Corollary 2.5(i) carries over. □

6.6. Lemma. *Let* $x \in W^J$, $a \in W^I$, $b \in W$. *Then* $[I, a, b] \leqslant [J, x, 1]$ *if and only if* $I \subseteq J$, $b \in W_J$, *and* $xb \leqslant a$.

Proof. If these conditions are satisfied, it is immediate that $[I, a, b] \leqslant_2 [J, x, 1]$, whence $[I, a, b] \leqslant [J, x, 1]$. Conversely, let this be the case and assume that

$$v_0 = [I, a, b], \quad v_1, \quad \ldots, \quad v_s = [J, x, 1]$$

is a sequence of elements of V such that for $i = 1, \ldots, s$ either $v_{i-1} \leqslant_1 v_i$ or $v_{i-1} \leqslant_2 v_i$. If $s = 1$ the condition of the lemma is easily seen to hold. So assume that $s > 1$. By induction we may assume that v_1 is of the form $[K, c, d]$, with $K \subseteq J$, $d \in W_J$, and $xd \leqslant c$. If $v_0 \leqslant_1 v_1$ then $xb \leqslant xd \leqslant c \leqslant a$. Since $d \in W_J$ and $b \leqslant d$, we have $b \in W_J$. Moreover, $I \subseteq K \subseteq J$. This shows that the conditions of the lemma hold.

If $v_0 \leqslant_2 v_1$, there is $z \in W_K$ with $b = dz$, $l(b) = l(d) + l(z)$, and $cz \leqslant a$. Then $xb = xdz \leqslant cz \leqslant a$. Moreover, $I \subseteq K \subseteq J$ and $b = dz \in W_J$, since $d \in W_J$ and $z \in W_K \subset W_J$. The lemma follows. □

The next lemma is a partial analog of Lemma 2.1(b).

6.7. Lemma. *Let $[I, x', w'] \leqslant [J, x, w]$ and assume that $s \in S$ is such that $sw > w$, $sw' > w'$. Then $[I, x', sw'] \leqslant [J, x, sw]$.*

Proof. Take a chain $v_0 = [I, x', w']$, $v_1, \ldots, v_r = [J, x, w]$ such that for $i = 1, \ldots, r$ either $v_{i-1} \leqslant_1 v_i$ or $v_{i-1} \leqslant_2 v_i$.

First, let $r = 1$. If $v_0 \leqslant_1 v_1$, it is immediate that $[I, x', sw'] \leqslant_1 [J, x, sw]$. If $v_0 \leqslant_2 v_1$ and z is as in the definition of $\leqslant_2$ (before Lemma 2.6) then z may also serve to show that $[I, x', sw'] \leqslant_2 [J, x, sw]$. Now let $r > 1$. Assume that $v_1 = [K, c, d]$. If $sd > d$, we may assume by induction that $[K, c, sd] \leqslant [J, x, sw]$. By the case $r = 1$ we know that $[I, x', sw'] \leqslant [K, c, sd]$, and the assertion follows.

If $sd < d$ and $v_0 \leqslant v_1$, we have $w' \leqslant d$. But then we must have $sw' \leqslant d$, whence

$$[I, x', sw'] \leqslant_1 [K, c, d] \leqslant [J, x, w] \leqslant_1 [J, x, sw].$$

If $sd < d$, we cannot have $v_0 \leqslant_2 v_1$. For if $w' = dz$, $l(w') = l(d) + l(z)$ then $sw' > w'$ implies $sd > d$.

We have proved the lemma. □

6.8. The map Δ. The results of Section 3 suggest the introduction of a map Δ, semilinear with respect to the automorphism of $\mathbb{Z}[u, u^{-1}]$ sending u to u^{-1}, such that for $v \in V$

$$\Delta(m_v) = u^{-2d(v)} \sum_{x \in V} b_{x,v} m_x,$$

satisfying

(A) $\Delta(e_t.m) = e_t^{-1}.\Delta(m)$ $(t \in \boldsymbol{S}, m \in \mathcal{N})$,
(B) $b_{[I,a,b],[J,1,1]} = (-1)^{l(a)+l(b)}(1 - u^2)^{|J-I|} R_{b,a}(u^2)$, for $a \in W^I$, $b \in W_J$, and $b \leqslant a$.

(A) is formula (7) and (B) is the particular case $x = 1$ of (14). The target space $\widetilde{\mathcal{M}}$ of Δ should be a completion of $\mathcal{M}$, as infinite sums arise.

6.9. Proposition.
(i) *There exists a unique Δ satisfying* (A) *and* (B).
(ii) $b_{w,v} = 0$ *if* $w \not\leqslant v$.
(iii) *If $w \leqslant v$ then $b_{w,v}$ is a polynomial in $\mathbb{Z}[u^2]$ with leading term $(-u^2)^{d(v)-d(w)}$.*

Proof. The formulas of Lemma 3.2 imply that

$$m_{J,x,w} = e^{-1}_{(x^{-1},1)}.e_{(1,w)}.m_{J,1,1}.$$

If Δ exists, it follows that

$$\Delta(m_{J,x,w}) = e_{(x,1)}.e^{-1}_{(1,w^{-1})}.\Delta(m_{J,1,1}), \tag{17}$$

showing that Δ is uniquely determined by (A) and (B).

We define Δ by (17). Then we have to prove (A).
If $s \in S$, $sw > w$, then by Lemma 3.2:

$$\begin{aligned}\Delta(e_{(1,s)}.m_{J,x,w}) &= \Delta(m_{J,x,sw}) = e_{(x,1)}.e^{-1}_{(1,w^{-1}s)}.\Delta(m_{J,1,1})\\ &= e^{-1}_{(1,s)}\Delta(m_{J,x,w}),\end{aligned}$$

establishing part of (A). The case that $sw < w$ is dealt with similarly, as well as the cases of $t = (s,1)$ with $sx \in W^J$.

There remains the case that $t = (s,1)$ and $sx = x\sigma$ with $\sigma \in W_J$. Then

$$\Delta\big(e(s,1).m_{J,x,w}\big) = \Delta(m_{J,x,w\sigma}) = e^{-1}_{(1,(w\sigma)^{-1})}.\Delta(m_{J,x,w}).$$

For (A) to hold this should be equal to $e^{-1}_{(s,1)}.\Delta(m_{J,x,w})$. Using (17) we see that it suffices to deal with the case that $w = 1$. In that case we have to prove

$$e_{(s,1)}.\Delta(m_{J,x,1}) = e_{(1,\sigma)}.\Delta(m_{J,x,1}).$$

Now the arguments of Remark 3.9 show that (B) implies the validity of (14) in the present situation. The equality to be proved then follows by using properties of R-polynomials, as in the proof of Lemma 3.8.

We prove (ii) by induction on $l(w)$. For $w = 1$ the assertion is true by Lemma 6.6. Let $w \neq 1$ and take $s \in S$ with $sw < w$. Formula (9) holds in our situation and implies

$$b_{[I,a,b],[J,x,w]} = \begin{cases} b_{[I,a,sb],[J,x,sw]} & \text{if } sb < b,\\ \big(1-u^2\big)b_{[I,a,b],[J,x,sw]} + u^2 b_{[I,a,sb],[J,x,sw]} & \text{if } sb > b.\end{cases}$$

If $b_{[I,a,b],[J,x,w]} \neq 0$ and $sb < b$ then we can conclude by induction that $[I,a,sb] \leqslant [J,x,sw]$. Application of Lemma 6.7 shows that $[I,a,b] \leqslant [J,x,w]$.

If $b_{[I,a,b],[J,x,w]} \neq 0$ and $sb > b$ then induction gives that $[I,a,b] \leqslant [J,x,sw]$ or $[I,a,sb] \leqslant [J,x,sw]$. In the first case we have $[J,x,sw] \leqslant_1 [J,x,w]$ and in the second case we must have $[I,a,b] \leqslant_1 [I,a,sb] \leqslant [J,x,sw] \leqslant_1 [J,x,w]$. In both cases $[I,a,b] \leqslant [J,x,w]$. This establishes (ii).

The proof of (iii) is like the proof of Proposition 3.5(i). □

6.10. From Lemma 6.5(ii) and Proposition 6.9(ii) we infer that Δ^2 is a module homomorphism $\mathcal{M} \to \widetilde{\mathcal{M}}$, given by

$$\Delta^2(m_v) = \sum_{w\in V}\Bigg(\sum_{w\leqslant z\leqslant v} u^{2d(v)-2d(z)} b_{w,z}\big(u^{-2}\big)b_{z,v}\big(u^2\big)\Bigg)m_w.$$

The results of Section 3 suggest the conjecture that $\Delta^2 = 1$. But so far I have not been able to prove this.

If $\Delta^2 = 1$, one can show that there exist Kazhdan–Lusztig polynomials $c_{w,v}$ with the properties of Proposition 4.4. In fact, the existence of such polynomials, for all $v, w \in V$, is equivalent with the involutive property of Δ.

Some support for the conjecture is provided by calculations made by W. van der Kallen. With his program for computing the Kazhdan–Lusztig polynomials of Section 4, he did some experimentation with formula (15) in the case of affine Weyl groups of small rank and in the case of some non-crystallographic Coxeter groups. The experiments produced polynomials $c_{w,v}$ with positive integral coefficients (see Appendix A).

It is natural to ask whether there is some geometric background to the constructions of the present section.

Acknowledgments

I am grateful to M. Brion and W. van der Kallen, for their interest in this paper and for their comments, and to the referee for pointing out an error.

Appendix A. (by Wilberd van der Kallen)

We computed the $c_{w,v}$ of Theorem 4.2 with a Mathematica program. We obtained them for all w, v when G is of rank two, and for $v = B = [D, 1, 1]$ when G is of rank three. After minor changes in the program, we could also explore a few affine Weyl groups and a few Coxeter groups that are not Weyl groups.

The set V of $B \times B$ orbits has size 1800 for type A_3 and it has size 7056 for type B_3. As memory becomes a problem we had to give up on computing and storing the entire partial order on V. But note that $\tilde{b}_{v,w} \neq 0$ exactly when $v \leqslant w$. Therefore, we replace in recursion (15) the sum

$$\sum_{w<y\leqslant v} \tilde{c}_{y,v}\tilde{b}_{w,y}$$

with a sum over y with $w \neq y$ and $\tilde{b}_{y,v} \neq 0$.

We also need an efficient solution of the word problem in our Weyl groups. We are primarily interested in very small Weyl groups, so we can simply start from a faithful representation. (Matrix coefficients have to lie in a ring for whose elements a normal form has been implemented.) We might have stored the multiplication table, but we prefer manipulating words. As Mathematica is very good at replacement rules in terms of pattern matching, we have it look for rules that make a word go down in length or lexicographic order. Five rules suffice for B_3. In the main computation these rules are then applied automatically when `reduce` is called. Then it is straightforward to introduce things like distinguished coset representatives, the length function, the R-polynomials. We use the R-polynomials also for storing the Bruhat order.

As the partial order on V is not readily available, we do not use directly that $\tilde{b}_{v,w} = 0$ when $v \nleqslant w$. Instead, we check if the criteria in Proposition 2.4, Corollary 2.5 guarantee vanishing. To be specific, suppose we want to compute $\tilde{b}_{v,w}$ with $v = [I, a, b]$ and $w = [J, x, y]$. We first check if $v = w$. Then we check if $I \subseteq J$, $a \geqslant x$, $l(b) - l(a) \leqslant l(y) - l(x)$. If not, then $\tilde{b}_{v,w}$ certainly vanishes. If $y = 1$ we check if $b \in W_J$. Apart from these checks, the procedure is as described in the paper: To compute $\tilde{b}_{v,w}$ one first uses (10) to reduce to $y = 1$, then one applies the formula in Lemma 3.8.

As we recompute $\tilde{b}_{v,w}$ each time they are needed, it should not be a surprise that the program is slow. It took about a week to compute the $c_{w,B}$ for type B_3. (Actually, at that time we did not yet use R-polynomials.) On a machine with more memory one could speed things up.

The Mathematica files are available on our web site. See `http://www.math.uu.nl/people/vdkallen/kallen.html`. There one also finds more of the output, some of it in PostScript, most of it in Mathematica InputForm.

A.1. Tables. We put $q = u^2$. In the tables we have left out all cases where $c_{w,v}$ equals zero or one.

Table 1
Type A_2, $c_{w,v}$ for $v = [D, 1, 1]$

$c_{w,v}$	w
$1+q$	$[\emptyset, s_1s_2, 1]$
$1+q$	$[\emptyset, s_1s_2, s_2]$
$1+q$	$[\emptyset, s_2s_1, 1]$
$1+q$	$[\emptyset, s_2s_1, s_1]$
$1+q$	$[\emptyset, s_1s_2s_1, s_1]$
$1+q$	$[\emptyset, s_1s_2s_1, s_2]$
$1+2q+q^2$	$[\emptyset, s_1s_2s_1, 1]$

Observe there is some duplication caused by the symmetry which interchanges s_1 with s_2.

Table 2
Type B_2, $c_{w,v}$ for $v = [D, 1, 1]$

$c_{w,v}$	w
$1+q$	$[\emptyset, s_1s_2, 1]$
$1+q$	$[\emptyset, s_1s_2, s_2]$
$1+q$	$[\emptyset, s_2s_1, 1]$
$1+q$	$[\emptyset, s_2s_1, s_1]$
$1+q$	$[\emptyset, s_1s_2s_1, s_2s_1]$
$1+q$	$[\emptyset, s_2s_1s_2, s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2, s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2, s_2s_1]$
$1+q$	$[\{1\}, s_2s_1s_2, 1]$
$1+q$	$[\{1\}, s_2s_1s_2, s_1]$
$1+q$	$[\{2\}, s_1s_2s_1, 1]$
$1+q$	$[\{2\}, s_1s_2s_1, s_2]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_2]$
$1+2q$	$[\emptyset, s_2s_1s_2, s_1]$
$1+2q$	$[\emptyset, s_2s_1s_2, s_2]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1, 1]$
$1+3q+q^2$	$[\emptyset, s_2s_1s_2, 1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_2]$
$1+4q+3q^2$	$[\emptyset, s_1s_2s_1s_2, 1]$

Table 3
Type G_2, $c_{w,v}$ for $v=[D,1,1]$

$c_{w,v}$	w
$1+q$	$[\emptyset, s_1s_2, 1]$
$1+q$	$[\emptyset, s_1s_2, s_2]$
$1+q$	$[\emptyset, s_2s_1, 1]$
$1+q$	$[\emptyset, s_2s_1, s_1]$
$1+q$	$[\emptyset, s_1s_2s_1, s_2s_1]$
$1+q$	$[\emptyset, s_2s_1s_2, s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2, s_2s_1s_2]$
$1+q$	$[\emptyset, s_2s_1s_2s_1, s_1s_2s_1]$
$1+q$	$[\emptyset, s_1s_2s_1s_2s_1, s_2s_1s_2s_1]$
$1+q$	$[\emptyset, s_2s_1s_2s_1s_2, s_1s_2s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_1s_2s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_2s_1s_2s_1]$
$1+q$	$[\{1\}, s_2s_1s_2, 1]$
$1+q$	$[\{1\}, s_2s_1s_2, s_1]$
$1+q$	$[\{1\}, s_1s_2s_1s_2, s_2]$
$1+q$	$[\{1\}, s_1s_2s_1s_2, s_2s_1]$
$1+q$	$[\{1\}, s_2s_1s_2s_1s_2, s_1s_2]$
$1+q$	$[\{1\}, s_2s_1s_2s_1s_2, s_1s_2s_1]$
$1+q$	$[\{2\}, s_1s_2s_1, 1]$
$1+q$	$[\{2\}, s_1s_2s_1, s_2]$
$1+q$	$[\{2\}, s_2s_1s_2s_1, s_1]$
$1+q$	$[\{2\}, s_2s_1s_2s_1, s_1s_2]$
$1+q$	$[\{2\}, s_1s_2s_1s_2s_1, s_2s_1]$
$1+q$	$[\{2\}, s_1s_2s_1s_2s_1, s_2s_1s_2]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_2]$
$1+2q$	$[\emptyset, s_2s_1s_2, s_1]$
$1+2q$	$[\emptyset, s_2s_1s_2, s_2]$
$1+2q$	$[\emptyset, s_1s_2s_1s_2, s_1s_2]$
$1+2q$	$[\emptyset, s_1s_2s_1s_2, s_2s_1]$
$1+2q$	$[\emptyset, s_2s_1s_2s_1, s_1s_2]$
$1+2q$	$[\emptyset, s_2s_1s_2s_1, s_2s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1s_2s_1, s_1s_2s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1s_2s_1, s_2s_1s_2]$
$1+2q$	$[\emptyset, s_2s_1s_2s_1s_2, s_1s_2s_1]$
$1+2q$	$[\emptyset, s_2s_1s_2s_1s_2, s_2s_1s_2]$

$c_{w,v}$	w
$1+2q$	$[\{1\}, s_1s_2s_1s_2, 1]$
$1+2q$	$[\{1\}, s_1s_2s_1s_2, s_1]$
$1+2q$	$[\{1\}, s_2s_1s_2s_1s_2, s_2]$
$1+2q$	$[\{1\}, s_2s_1s_2s_1s_2, s_2s_1]$
$1+2q$	$[\{2\}, s_2s_1s_2s_1, 1]$
$1+2q$	$[\{2\}, s_2s_1s_2s_1, s_2]$
$1+2q$	$[\{2\}, s_1s_2s_1s_2s_1, s_1]$
$1+2q$	$[\{2\}, s_1s_2s_1s_2s_1, s_1s_2]$
$1+3q$	$[\{1\}, s_2s_1s_2s_1s_2, 1]$
$1+3q$	$[\{1\}, s_2s_1s_2s_1s_2, s_1]$
$1+3q$	$[\{2\}, s_1s_2s_1s_2s_1, 1]$
$1+3q$	$[\{2\}, s_1s_2s_1s_2s_1, s_2]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1, 1]$
$1+3q+q^2$	$[\emptyset, s_2s_1s_2, 1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_1s_2s_1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_2s_1s_2]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_1]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_2]$
$1+4q+q^2$	$[\emptyset, s_2s_1s_2s_1, s_1]$
$1+4q+q^2$	$[\emptyset, s_2s_1s_2s_1, s_2]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_1s_2]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_2s_1]$
$1+4q+q^2$	$[\emptyset, s_2s_1s_2s_1s_2, s_1s_2]$
$1+4q+q^2$	$[\emptyset, s_2s_1s_2s_1s_2, s_2s_1]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2, 1]$
$1+5q+3q^2$	$[\emptyset, s_2s_1s_2s_1, 1]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_1s_2]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_2s_1]$
$1+6q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_1]$
$1+6q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_2]$
$1+6q+3q^2$	$[\emptyset, s_2s_1s_2s_1s_2, s_1]$
$1+6q+3q^2$	$[\emptyset, s_2s_1s_2s_1s_2, s_2]$
$1+7q+5q^2$	$[\emptyset, s_1s_2s_1s_2s_1, 1]$
$1+7q+5q^2$	$[\emptyset, s_2s_1s_2s_1s_2, 1]$
$1+7q+5q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_1]$
$1+7q+5q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, s_2]$
$1+8q+7q^2$	$[\emptyset, s_1s_2s_1s_2s_1s_2, 1]$

Table 4
Type A_3, sample of large $c_{w,v}$ for $v=[D,1,1]$

$c_{w,v}$	w
$1+7q+12q^2+4q^3$	$[\emptyset, s_1s_2s_1s_3s_2s_1, s_2]$
$1+7q+12q^2+4q^3$	$[\emptyset, s_1s_2s_3s_2s_1, 1]$
$1+7q+13q^2+4q^3$	$[\emptyset, s_1s_2s_1s_3s_2s_1, s_1]$
$1+7q+13q^2+4q^3$	$[\emptyset, s_1s_2s_1s_3s_2s_1, s_3]$
$1+7q+13q^2+4q^3$	$[\emptyset, s_1s_2s_1s_3s_2, 1]$
$1+7q+13q^2+4q^3$	$[\emptyset, s_2s_1s_3s_2s_1, 1]$
$1+8q+19q^2+10q^3$	$[\emptyset, s_1s_2s_1s_3s_2s_1, 1]$

Table 5
Type B_3, sample of large $c_{w,v}$ for $v=[D,1,1]$

$c_{w,v}$	w
$1+18q+71q^2+73q^3+11q^4$	$[\emptyset, s_2s_1s_3s_2s_1s_3s_2s_3, s_3]$
$1+18q+71q^2+73q^3+11q^4$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2, s_1]$
$1+18q+74q^2+75q^3+11q^4$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2, s_3]$
$1+18q+74q^2+76q^3+13q^4$	$[\emptyset, s_1s_2s_3s_2s_1s_3s_2s_3, s_2]$
$1+18q+71q^2+78q^3+15q^4$	$[\emptyset, s_2s_1s_3s_2s_1s_3s_2s_3, s_2]$
$1+18q+71q^2+78q^3+15q^4$	$[\emptyset, s_1s_2s_3s_2s_1s_3s_2s_3, s_1]$
$1+18q+72q^2+79q^3+15q^4$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2s_3, s_1s_3]$
$1+18q+72q^2+79q^3+15q^4$	$[\emptyset, s_2s_1s_3s_2s_1s_3s_2, 1]$
$1+18q+75q^2+81q^3+16q^4$	$[\emptyset, s_1s_2s_3s_2s_1s_3s_2s_3, s_3]$
$1+18q+75q^2+81q^3+16q^4$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2, s_2]$
$1+19q+81q^2+107q^3+29q^4$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2s_3, s_1]$
$1+19q+81q^2+107q^3+29q^4$	$[\emptyset, s_2s_1s_3s_2s_1s_3s_2s_3, 1]$
$1+19q+85q^2+113q^3+34q^4+q^5$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2s_3, s_2]$
$1+19q+85q^2+113q^3+34q^4+q^5$	$[\emptyset, s_1s_2s_3s_2s_1s_3s_2s_3, 1]$
$1+19q+86q^2+116q^3+36q^4+2q^5$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2s_3, s_3]$
$1+19q+86q^2+116q^3+36q^4+2q^5$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2, 1]$
$1+20q+96q^2+153q^3+67q^4+6q^5$	$[\emptyset, s_1s_2s_1s_3s_2s_1s_3s_2s_3, 1]$

A.2. Poincaré polynomials for intersection cohomology.

Put $\mathcal{IP}_v(q)=\sum_{i\geqslant 0}\dim IH^{2i}(\bar{v})\,q^i$.

Table 6
Poincaré polynomials $\mathcal{IP}_v$ for type A_1

$\mathcal{IP}_v$	v
1	$[\emptyset, s_1, 1]$
$1+q$	$[\emptyset, 1, 1]$
$1+q$	$[\emptyset, s_1, s_1]$
$1+q+q^2$	$[\{1\}, 1, 1]$
$1+2q+q^2$	$[\emptyset, 1, s_1]$
$1+q+q^2+q^3$	$[\{1\}, 1, s_1]$

Table 7
Poincaré polynomials $\mathcal{IP}_v$ for $v = [D, 1, 1]$

$\mathcal{IP}_v$	Type
$1 + q + q^2$	A_1
$1 + 4q + 9q^2 + 9q^3 + 4q^4 + q^5$	A_2
$1 + 6q + 17q^2 + 24q^3 + 17q^4 + 6q^5 + q^6$	B_2
$1 + 10q + 33q^2 + 64q^3 + 80q^4 + 64q^5 + 33q^6 + 10q^7 + q^8$	G_2
$1 + 11q + 56q^2 + 154q^3 + 250q^4 + 250q^5 + 154q^6 + 56q^7 + 11q^8 + q^9$	A_3
$1 + 23q + 181q^2 + 770q^3 + 2046q^4 + 3610q^5 + 4350q^6 + 3610q^7 + 2046q^8 + 770q^9 + 181q^{10} + 23q^{11} + q^{12}$	B_3

A.3. Experiments with other Coxeter groups.

Table 8
Dihedral group of order 10, $c_{w,v}$ for $v = [D, 1, 1]$

$c_{w,v}$	w
$1+q$	$[\emptyset, s_1s_2, 1]$
$1+q$	$[\emptyset, s_1s_2, s_2]$
$1+q$	$[\emptyset, s_1s_2s_1, s_2s_1]$
$1+q$	$[\emptyset, s_1s_2s_1s_2, s_2s_1s_2]$
$1+q$	$[\emptyset, s_1s_2s_1s_2s_1, s_1s_2s_1]$
$1+q$	$[\emptyset, s_1s_2s_1s_2s_1, s_2s_1s_2]$
$1+q$	$[\{1\}, s_2s_1s_2, 1]$
$1+q$	$[\{1\}, s_2s_1s_2, s_1]$
$1+q$	$[\{1\}, s_1s_2s_1s_2, s_2]$
$1+q$	$[\{1\}, s_1s_2s_1s_2, s_2s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_1]$
$1+2q$	$[\emptyset, s_1s_2s_1, s_2]$

$c_{w,v}$	w
$1+2q$	$[\emptyset, s_1s_2s_1s_2, s_1s_2]$
$1+2q$	$[\emptyset, s_1s_2s_1s_2, s_2s_1]$
$1+2q$	$[\{1\}, s_1s_2s_1s_2, 1]$
$1+2q$	$[\{1\}, s_1s_2s_1s_2, s_1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1, 1]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_1s_2]$
$1+3q+q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_2s_1]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_1]$
$1+4q+q^2$	$[\emptyset, s_1s_2s_1s_2, s_2]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2, 1]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_1]$
$1+5q+3q^2$	$[\emptyset, s_1s_2s_1s_2s_1, s_2]$
$1+6q+5q^2$	$[\emptyset, s_1s_2s_1s_2s_1, 1]$

As usual, we deleted all cases where $c_{w,v}$ equals zero or one. We also removed all duplication caused by the symmetry which interchanges s_1 with s_2 and thus $\{1\}$ with $\{2\}$.

Table 9
Affine type A_1, a sample

$c_{w,v}$	w	v
$1+4q+3q^2+q^3$	$[\emptyset, s_2s_1s_2s_1, 1]$	$[\{1,2\}, 1, s_1s_2s_1]$
$1+4q+3q^2+q^3$	$[\emptyset, s_2s_1s_2s_1, s_1]$	$[\{1,2\}, 1, s_1s_2s_1]$
$1+4q+3q^2+2q^3$	$[\emptyset, s_1s_2s_1s_2, 1]$	$[\{1,2\}, 1, s_1s_2s_1s_2]$
$1+4q+3q^2+2q^3$	$[\emptyset, s_1s_2s_1s_2, s_1]$	$[\{1,2\}, 1, s_1s_2s_1s_2]$
$1+4q+3q^2+2q^3$	$[\emptyset, s_1s_2s_1s_2, s_2]$	$[\{1,2\}, 1, s_1s_2s_1s_2]$
$1+4q+3q^2+2q^3$	$[\emptyset, s_1s_2s_1s_2, s_1s_2]$	$[\{1,2\}, 1, s_1s_2s_1s_2]$

We restricted the lengths of elements of W to four and looked for large $c_{w,v}$. Again we removed duplicates. One recovers them by interchanging s_1 with s_2.

Table 10
Affine type A_2, sample of large $c_{w,v}$

$c_{w,v}$	w	v
$1+6q+5q^2$	$[\emptyset, s_2s_1s_3, s_1]$	$[\{1,2\}, 1, s_1s_2s_3]$
$1+3q+3q^2+q^3$	$[\emptyset, s_3s_2s_1, 1]$	$[\{1,2,3\}, 1, s_1s_3s_2]$
$1+3q+3q^2+q^3$	$[\emptyset, s_3s_2s_1, s_1]$	$[\{1,2,3\}, 1, s_1s_3s_2]$

We restricted the lengths of elements of W to three.

Table 11
Type H_3, sample of intermediate size $c_{w,v}$ for $v=[D,1,1]$

$c_{w,v}$	w
$1+21q+62q^2+33q^3$	$[\{1\}, s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_1s_2s_1s_2s_1]$
$1+23q+65q^2+33q^3$	$[\{3\}, s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_2s_3s_2]$
$1+23q+65q^2+33q^3$	$[\{3\}, s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2s_1, s_2s_1s_2s_3s_2s_1]$
$1+22q+69q^2+33q^3$	$[\emptyset, s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3, s_1s_2s_3s_2s_1]$
$1+22q+69q^2+33q^3$	$[\emptyset, s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_2s_3s_2s_1s_2]$
$1+24q+70q^2+34q^3$	$[\{3\}, s_1s_2s_3s_2s_1s_2s_1s_3s_2s_1, s_2s_3s_2s_1]$
$1+24q+70q^2+34q^3$	$[\{3\}, s_1s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_2s_1s_2s_3s_2]$
$1+26q+85q^2+40q^3$	$[\emptyset, s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_1s_2s_1s_2s_1]$
$1+26q+85q^2+40q^3$	$[\emptyset, s_3s_2s_1s_2s_1s_3s_2s_1s_2s_3, s_1s_2s_3s_2s_1]$
$1+26q+81q^2+44q^3$	$[\{3\}, s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2, s_1s_2s_3s_2]$
$1+26q+81q^2+44q^3$	$[\{3\}, s_2s_1s_2s_3s_2s_1s_2s_1s_3s_2s_1, s_1s_2s_3s_2s_1]$

For some w the computation of $c_{w,v}$ was not feasible on our machine. Therefore, we just present a few $c_{w,v}$ that were still within reach. (It gets more difficult as $\dim(v)-\dim(w)$ increases.)

Note added in proof

In a recent preprint by Y. Chen and M. Dyer (On the combinatorics of $B\times B$-orbits on group compactifications, J. Algebra, in press) it is shown that the Bruhat order of Section 2 can be understood in the context of the “twisted Bruhat orders” on Coxeter groups, introduced by M. Dyer. The Coxeter groups which appear here are in general neither finite nor affine. The authors also prove the conjecture made in Section 6.10.

References

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1982).
[B1] M. Brion, The behaviour at infinity of the Bruhat decomposition, Comm. Math. Helv. 73 (1998) 137–174.
[B2] M. Brion, Rational smoothness and fixed points of torus actions, Transform. Groups 4 (1999) 127–156.
[BJ] M. Brion, R. Joshua, Vanishing of odd intersection cohomology II, preprint.
[BT] A. Borel, J. Tits, Compléments à l’article “Groupes réductifs”, Publ. Math. Inst. Hautes Études Sci. 41 (1972) 253–276.

[C] R.W. Carter, Finite Groups of Lie Type. Conjugacy Classes and Complex Characters, Wiley, 1985.
[DP] C. De Concini, C. Procesi, Complete symmetric varieties, in: Invariant Theory, in: Lecture Notes in Math., Vol. 996, Springer, 1983, pp. 1–44.
[DS] C. De Concini, T.A. Springer, Compactification of symmetric varieties, Transform. Groups 4 (1999) 273–300.
[GJ] O. Gabber, A. Joseph, Towards the Kazhdan–Lusztig conjecture, Ann. Sci. École. Norm. Sup. 14 (1981) 261–302.
[K] F. Knop, On the set of orbits for a Borel group, Comm. Math. Helv. 70 (1995) 285–309.
[KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979) 165–184.
[LV] G. Lusztig, D.A. Vogan, Singularities of closures of K-orbits on flag manifolds, Invent. Math. 71 (1983) 365–379.
[MS] J.G.M. Mars, T.A. Springer, Hecke algebra representations related to spherical varieties, J. Represent. Theory 2 (1998) 33–69.
[RS] R.W. Richardson, T.A. Springer, The Bruhat order on symmetric varieties, Geom. Dedicata 35 (1990) 389–436.
[S] T.A. Springer, Combinatorics of B-orbit closures in a wonderful compactification, in: Algebraic Groups and Arithmetic (Mumbai, 2001), Tata Inst. Fund. Res. Stud. Math., in press.