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Longest Weight Vectors and Excellent Filtrations

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Let B be a Borel subgroup in a connected simply connected semisimple algebraic group G defined over an algebraically closed field k. We consider several homological properties of B modules in relation with the existence of certain filtrations of such a module. In particular we show that the notion of an excellent filtration (see Polo [P]) leads to an example of a highest weight category in the sense of Cline, Parshall, Scott [CPS]. Another example, in some sense dual, is also treated. (1.6(ii)).

§1

1.1. *Conventions.* For unexplained notations, terminology etc. we refer to [J]. Fix a maximal torus T in B and a Weyl group invariant inner product $(\,,\,)$ on $X(T)\otimes_{\mathbb{Z}}\mathbb{R}$. For a weight λ, we call (λ, λ) its length. A weight vector v in a B module M is called a *longest weight vector* in M if all other weights of the B module generated by v are strictly shorter than the weight of v. Thus a highest weight vector (i.e. a vector that generates a one dimensional B module) is an example of a longest weight vector. Long M denotes the T module spanned by the longest weight vectors in M and $\mathrm{Long}_\lambda\, M$ denotes the λ weight space of Long M. Thus Long is a functor from B modules to T modules.

1.2. In Jantzen's book [J] the weights λ for which $\mathrm{ind}_B^G \lambda$ is not zero are called dominant. Here we follow [P] instead and write P^- for the set of these weights. Put $P^+=\{\lambda\in X(T)\mid -\lambda\in P^-\}$. We order the weights according to what we call the *excellent order*: $\lambda\leqq\mu$ if either $(\lambda, \lambda)<(\mu, \mu)$ or $\lambda=wv$, $\mu=zv$ for some $v\in P^-$, $w,z\in W$ with $w\leqq z$ (in the Bruhat order on W). Compare [De, Lemma 3.5]. We also define an *antipodal excellent order* for which $\lambda\leqq\mu$ means that $-\lambda\leqq-\mu$ in the excellent order. If π is a set of weights then we say that π is *filled up* ("convenablement rempli" in [P], cf. "saturated" in [D]) if it is an ideal for the excellent order, i.e. if $\lambda\leqq\mu$ and $\mu\in\pi$ together imply $\lambda\in\pi$. We say that π is *round* if there is some $R\geqq 0$ such that $\pi=\{\lambda\in X(T)\mid(\lambda, \lambda)\leqq R\}$. Note that if π is round, it is also filled up.

1.3. If Ω is a subset of W then X_Ω denotes the union (with reduced subscheme structure) of the Schubert varieties X_w in G/B with $w\in\Omega$. (We write X_w where

[J] writes $X(w)$). If M is a B module, $H_w(M)$ denotes the B module $\Gamma(X_w, \mathscr{L}(M))$ where $\mathscr{L}(M)$ is the usual vector bundle [J, I 5.8]. Similarly $H_\Omega(M)$ denotes $\Gamma(X_\Omega, \mathscr{L}(M))$. Let $w \in W$. By ∂X_w we denote X_Ω where $\Omega = \{z \in W \mid z < w\}$. If λ is a weight, choose $\lambda^- \in P^- \cap W\lambda$ and $w \in W$ such that $\lambda = w\lambda^-$ and such that w is minimal with this property. (Given λ, the choice of λ^- and w is unique [BLIE, Ch. 5 §6 Cor.; Ch. 4 §1 Ex. 3]). Then put $P(\lambda) = H_w(\lambda^-)$, a dual Joseph module, cf. [P], and $Q(\lambda) = \ker(P(\lambda) \to \Gamma(\partial X_w, \mathscr{L}(\lambda^-)))$, a "minimal relative Schubert module". These modules $P(\lambda)$, $Q(\lambda)$ will be building blocks in what follows. Both $P(\lambda)$ and $Q(\lambda)$ have one dimensional socles of weight λ. (Combine the proof of [J, Prop. II 2.2] with [J, II 13.3]).

1.4. If E is a B module with $\mathrm{Hom}_B(E, E)$ one dimensional, then we say that a B module M is *isotypical* of type E if M is a direct sum of copies of E. (The number of copies may be zero, finite, infinite). Equivalently, M is isotypical of type E if the evaluation map $\mathrm{Hom}_B(E, M) \otimes_k E \to M$ is an isomorphism. An *excellent filtration* of a B module M is a sequence of submodules $0 = F_0 \subseteq F_1 M \subseteq F_2 M \subseteq \ldots$ such that

(i) M is the union of the $F_i M$.

(ii) For each $i \geqq 1$ there is a weight λ so that $F_i M / F_{i-1} M$ is isotypical of type $P(\lambda)$.

Replacing $P(\lambda)$ by $Q(\lambda)$ one gets the definition of a *relative Schubert filtration* of M. Observe that our definition of an excellent filtration does not quite agree with the literature [P]. This is because we want to allow arbitrarily large modules, not just those of countable dimension, and also because we prefer "canonical filtrations" as in [F, Th. 4]. For a B module M of countable dimension (finite or infinite) one may argue as in [F] to see that if M has an excellent filtration it also has one in which the $F_i M / F_{i-1} M$ are indecomposable (or zero). That is what is customarily required in definitions of this type (cf. good filtrations, Weyl filtrations, Joseph filtrations, Schubert filtrations).

1.5. Let π be a set of weights. As in [CPS] we denote by $\mathscr{C}[\pi]$ the category of all B modules all of whose weights are in π. The right adjoint of the embedding of $\mathscr{C}[\pi]$ into the category $\mathscr{C}$ of all B modules is known as O_π, cf. [D]. Thus $O_\pi M$ is the largest B submodule of M whose weights are all in π. Now recall that a highest weight category structure on $\mathscr{C}$ consists of a partially ordered index set Λ and a family $\{A(\lambda)\}_{\lambda \in \Lambda}$ of B-modules with $A(\lambda)$ having (irreducible) socle $S(\lambda)$ and injective hull $I(\lambda)$, such that certain axioms [CPS, 3.1] are satisfied. We now list our theorems.

1.6. **Theorem.** (i) *Endow the set* $\Lambda = X(T)$ *of weights with the excellent order* (1.2) *and choose for* $A(\lambda)$ *the dual Joseph module* $P(\lambda)$ *with socle* λ. (1.3). *Then this defines a highest weight category structure on the category of B modules.*

(ii) *Now endow* Λ *with the antipodal excellent order* (1.2) *and choose for* $A(\lambda)$ *the minimal relative Schubert module* $Q(\lambda)$ *of* 1.3. *Again this defines a highest weight category structure on the category of B modules.*

1.7. **Theorem.** (a) *Let M be a B module with excellent filtration.*

(i) *M is acyclic for Long*

(ii) *For every module N with relative Schubert filtration, the B module $M \otimes N$ is B acyclic, i.e. $H^i(B, M \otimes N) = 0$ for $i > 0$.*

(iii) *If π is filled up* (1.2) *then M is acyclic for O_π and $O_\pi M$ has an excellent filtration* (*cf.* [D]).

(b) *Let M be a B module. Assume that one of the following is satisfied.*

(i) *R^1 Long $M = 0$,*

(ii) *For every module N with relative Schubert filtration $H^1(B, M \otimes N) = 0$,*

(iii) *If π is filled up then $R^1 O_\pi(M) = 0$,* (*cf.* [D]).

Then M has an excellent filtration.

1.8. **Corollary.** (i) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of B modules and M, M' have excellent filtrations, then so does M''.*

(ii) *If M is an inductive limit of modules with excellent filtration, then M itself has an excellent filtration* (*cf.* 2.1).

1.9. **Theorem.** (a) *Let M have a relative Schubert filtration.*

(i) *If π is round* (1.2) *then M is acyclic for O_π and $O_\pi M$ has a relative Schubert filtration* (*cf.* [D]).

(ii) *If $w \in W$ then M is acyclic for H_w.*

(b) *Let M be a B module. Assume that one of the following is satisfied.*

(i) *For every module N with excellent filtration $H^1(B, M \otimes N) = 0$,*

(ii) *If π is round then $R^1 O_\pi(M) = 0$.*

Then M has a relative Schubert filtration.

1.10. Of course 1.9 has a corollary that is analogous to 1.8.

1.11. Let Ch denote formal character. (Unlike [J] we allow a formal character to be an infinite sum.)

Theorem. *Let M be a B module whose weight spaces are finite dimensional. Then*

$$Ch(M) \leqq \sum_\lambda \dim(\mathrm{Long}_\lambda(M))\, Ch(P(\lambda))$$

and equality holds if and only if M has an excellent filtration (*cf.* 3.8).

1.12. *Remark.* In particular, if M is a G module, this gives a criterion for the existence of a good filtration. For a G module any longest weight vector is of course a highest vector. To prove that a G module M with a given character has a good filtration, it thus suffices to show that there are not too many highest weight vectors, i.e. that the B socle of M has the same T module structure as in some module in characteristic zero with the same character as M.

1.13. **Theorem.** *The affine algebra $k[B]$ of B has a $B \times B$ module filtration which is an excellent filtration when restricted to $B \times 1$ and a relative Schubert filtration when restricted to $1 \times B$. The layers in the filtration are the exterior tensor products $P(-\lambda) \boxtimes Q(\lambda)$. Each occurs once.*

1.14. *Remark.* If λ is a weight and $\pi=\{\mu|(\mu,\mu)\leqq(\lambda,\lambda)\}$, then the Weyl group orbit $W\lambda$ is a coideal in π for the excellent order. With this coideal corresponds a derived category [CPS, Thm. 3.9]. It encodes the homological algebra of (the incidence algebra of) a partially ordered set (Bruhat order). Contemplating this, one invents the notion of a longest weight vector and the special choice of S_j in the proof of the key Theorem 2.12.

§ 2. Excellent Filtration of Injectives

2.1. Functors such as H_w and O_π are left exact and commute with inductive limits. If $M=\varinjlim M_i$ is an inductive limit of B modules M_i then its Hochschild injective resolution $M\to M\otimes k[B]\to M\otimes k[B]\otimes k[B]\to\dots$ is an inductive limit of resolutions of the M_i. Therefore the higher derived functors $R^jH_w=H_w^j$ and R^jO_π also commute with inductive limits (over filtered partially ordered index sets), and thus they commute with infinite direct sums. Recall that the fact that inductive limits of injectives are injective is intimately related with the fact that B modules are unions of noetherian submodules [BALG, Ch. X § 1 Ex. 21].

2.2. The following lemma relies on results of Ramanathan, obtained with the method of Frobenius splittings. It also explains why we call $Q(\lambda)$ a minimal relative Schubert module.

Lemma. *Let S_1, S_2 be B invariant reduced closed subschemes of G/B with $S_1\subseteq S_2$. (Thus S_i is a union of Schubert cells.) Let $\lambda\in P^-$. Then $\Gamma(S_i,\mathscr{L}(\lambda))$ and $\ker(\Gamma(S_2,\mathscr{L}(\lambda))\to\Gamma(S_1,\mathscr{L}(\lambda)))$ have relative Schubert filtrations. Thus a module with excellent filtration also has a relative Schubert filtration.*

Proof. Recall [R, Th. 2] that $\Gamma(S_2,\mathscr{L}(\lambda))\to\Gamma(S_1,\mathscr{L}(\lambda))$ is surjective. If $S_1\subseteq S_2\subseteq S_3$, write K_{ij} for $\ker(\Gamma(S_i,\mathscr{L}(\lambda))\to\Gamma(S_j,\mathscr{L}(\lambda)))$ when $1\leqq j<i\leqq 3$. The sequence $0\to K_{32}\to K_{31}\to K_{21}\to 0$ is exact. Therefore it suffices to prove the lemma for the case of K_{21} when S_1 is maximal among the B invariant reduced closed subschemes of S_2, distinct from S_2. (S_1 may be empty.) We claim that in this case K_{21} is actually one of our minimal relative Schubert modules $Q(\lambda)$, or zero. Namely, let X_w be the unique Schubert variety which is contained in S_2 but not in S_1. By [R, Th. 3] the scheme theoretic intersection of S_1 with X_w is reduced and must thus be ∂X_w. But the union of X_w with S_1 equals S_2, so any element of $Q=\ker(\Gamma(X_w,\mathscr{L}(\lambda))\to\Gamma(\partial X_w,\mathscr{L}(\lambda)))$ extends by zero to an element of K_{21}. This shows that the restriction map $K_{21}\to Q$ is bijective. Put $\mu=w\lambda$. If w is minimal among the $z\in W$ with $\mu=z\lambda$, then $Q=Q(\mu)$ by definition. If not, choose $z<w$ with $z\lambda=\mu$ and observe that Q is a submodule of the kernel of the surjection $H_w(\lambda)\to H_z(\lambda)$. Thus Q is zero by the following lemma.

2.3. **Lemma.** *For $\lambda\in P^-$, $w\in W$, there is a natural isomorphism $H_w(\lambda)\cong P(w\lambda)$.*

Proof. Polo's universal property [P, Cor. 2.4] of $H_w(\lambda)$ depends only on $w\lambda$, not w. (That is why we introduced the $P(\mu)$ notation).

2.4. Let us elaborate on Polo's universal property, which was the starting point for our investigations. Let λ be a weight. As $P(-\lambda)$ has socle $-\lambda$, $P(-\lambda)^*$ is generated as a B module by a weight vector e_λ of weight λ. In fact $P(-\lambda)^*$ is the Joseph-Demazure module generated by e_λ in the Weyl module whose highest weight lies in $W\lambda$ [P, 1.4].

Proposition (Polo, [P, 2.1, 2.4]). *Let λ be a weight and M a B module all whose weights μ satisfy $(\mu, \mu) \leqq (\lambda, \lambda)$. Then $f \mapsto f(e_\lambda)$ defines an isomorphism of vector spaces*

$$\mathrm{Hom}_B(P(-\lambda)^*, M) \to M_\lambda.$$

Dually,

$$\mathrm{Hom}_B(M, P(\lambda)) \cong (M_\lambda)^*.$$

Proof. For the dual statement, view M as an inductive limit of finite dimensional submodules N and use

$$\mathrm{Hom}_B(N, P(\lambda)) \cong \mathrm{Hom}_B(P(\lambda)^*, N^*).$$

2.5. Let $R = (\lambda, \lambda)$ and write $\mathscr{C}[R]$ for $\mathscr{C}[\pi]$ where π consists of the weights μ with $(\mu, \mu) \leqq R$.

Corollary (Polo, [P, Cor. 2.5]). *In $\mathscr{C}[R]$ the module $P(\lambda)^*$ is projective and the module $P(\lambda)$ is the injective hull of λ in $\mathscr{C}[R]$.*

2.6. **Lemma.** *Let λ, μ be weights of the same length. Then* $\mathrm{Hom}_B(P(\lambda), P(\mu))$, *which is the dual of $P(\lambda)_\mu$ by 2.4, is non-zero if and only if $\mu \leqq \lambda$ (in the excellent order).*

Proof. If $\mu \leqq \lambda$ then the restriction map $P(\lambda) \to P(\mu)$ is surjective. Conversely, suppose $\phi\colon P(\lambda) \to P(\mu)$ is non-zero. Then μ is a weight of $P(\lambda)$ so that $\mu \in W\lambda$. Choose $v \in P^- \cap W\lambda$ and $w, z \in W$ minimal so that $wv = \lambda$, $zv = \mu$. We claim that $z \leqq w$. If not, consider a non-zero element s of the socle of $P(\mu)$. As s is in the kernel of $P(\mu) \to P(\xi)$ for $\xi < \mu$, $(\xi, \xi) = (\mu, \mu)$, we see as in 2.2 that s extends by zero on X_w. That lifts s to a non-zero element of

$$\ker(\psi\colon \Gamma(X_z \cup X_w, \mathscr{L}(v)) \to \Gamma(X_w, \mathscr{L}(v)))$$

and the lifted element also has weight μ. But the weight space $\Gamma(X_z \cup X_w, \mathscr{L}(v))_\mu$ is one dimensional (use [R, Th. 2]), and ψ is surjective, so $P(\lambda)_\mu = 0$. Contradiction. For another proof that $z \leqq w$, one may dualize [Jo, 2.12], cf. [P, 1.4].

2.7. Denote the image of e_λ under $P(-\lambda)^* \to Q(-\lambda)^*$ also by e_λ.

Proposition. *Let λ be a weight and M a B module. Then $f \mapsto f(e_\lambda)$ defines an isomorphism* $\mathrm{Hom}_B(Q(-\lambda)^*, M) \to \mathrm{Long}_\lambda M$.

Proof. That $f(e_\lambda)$ is a longest weight vector in M follows from the fact that e_λ is one in $Q(-\lambda)^*$. Namely, if μ is a weight of $Q(-\lambda)^*$ of the same length as λ (longer is of course impossible), then $-\mu$ is a weight of $P(-\lambda)$, so we may choose $v \in P^-$, w, z minimal in W such that $wv = -\lambda$, $zv = -\mu$, and find $z \leqq w$. (See proof of 2.6.) If $\mu \neq \lambda$, then the composite of the surjections

$Q(-\lambda)_{-\mu} \xrightarrow{\simeq} P(-\lambda)_{-\mu} \to \Gamma(\partial X_w, \mathscr{L}(v))_{-\mu} \to P(-\mu)_{-\mu}$ vanishes, which is impossible.

Conversely, if $v \in \mathrm{Long}_\lambda M$, let ϕ: $P(-\lambda)^* \to M$ be the corresponding map and let N denote its image. (N is the B module generated by v.) We must show that ϕ factors through $Q(-\lambda)^*$. From the proof of 2.2 it is clear that $(P(-\lambda)/Q(-\lambda))^*$ has a filtration with layers $Q(-\mu)^*$ where $\mu \in W\lambda$, $\mu \neq \lambda$ (and in fact $\mu > \lambda$). Now $N_\mu = 0$ for such μ and $Q(-\mu)^*$ is generated by e_μ, so ϕ factors.

2.8. Now that we know that μ is the unique longest weight of $Q(\mu)$, we can be a little more specific about 2.2.

Corollary. *In the relative Schubert filtration of* $K_{21} = \ker(\Gamma(S_2, \mathscr{L}(\lambda)) \to \Gamma(S_1, \mathscr{L}(\lambda)))$ *in 2.2 a layer* $Q(\mu)$ *occurs if and only if* $\mu \in W\lambda$ *and* μ *is a weight of* K_{21}. *Its multiplicity is then one.*

2.9. **Corollary.** *Let* λ *be a weight and let* M *be a* B *module with* $(\mu, \mu) < (\lambda, \lambda)$ *for each weight* μ *of* M. *Then* $\mathrm{Ext}^1_B(Q(-\lambda)^*, M) = 0$.

Proof. If $I(M)$ is an injective hull of M then $\mathrm{Long}_\lambda(I(M)) \to \mathrm{Long}_\lambda(I(M)/M)$ is clearly surjective.

2.10. **Lemma** (Compare [P, Cor. 2.6]). *If* λ, μ *are weights, then* $H^1(B, Q(\lambda) \otimes P(\mu))$ *vanishes.*

Proof. If $(\lambda, \lambda) \leqq (\mu, \mu)$ then $H^1(B, Q(\lambda) \otimes P(\mu)) = \mathrm{Ext}^1_B(Q(\lambda)^*, P(\mu))$ vanishes because of 2.5. (The category $\mathscr{C}[R]$ is closed under extensions and 2.5 tells that extensions of $Q(\lambda)^*$ by $P(\mu)$ split). If $(\lambda, \lambda) > (\mu, \mu)$ then the extensions split because of 2.9.

2.11. **Lemma.** *Let* $w \in W$ *and let* M *be a* B *module with excellent filtration* $(F_i M)$. *Then* $F_i H_w(M) = H_w(F_i M)$ *defines an excellent filtration of* $H_w(M)$.

Proof (Extracted from [A]). Recall [P, Prop. 1.4.2] that there is an associative operation $*$ on W such that $H_w \circ H_z = H_{w*z}$ and such that $w*z = wz$ if $l(w) + l(z) = l(wz)$.

We may thus assume that w is a simple reflection y. For any weight μ there is (by [J, Prop. II 14.15(e)]) a surjection $H_y(P(\mu)) \to P(\mu)$ and this implies that $P(\mu)$ is acyclic for H_y, by [J, II 14.2, Prop. I 5.12]. By induction on i one sees that $F_i M$ is acyclic for H_y and the rest is clear.

2.12. Our key result is

Theorem. *Let* N *be an injective* B *module. Then* N *has an excellent filtration.*

Remarks. This is part of Theorem 1.6 (i). As Polo has reproduced an earlier draft of the proof faithfully [P], the reader may now see how much the text has deteriorated since.

2.13. *Proof.* Order the weights linearly, say $\xi_1, \xi_2, \ldots$ such that $i \leqq j$ if $\xi_i \leqq \xi_j$ in the excellent order. In other words, initial segments of the sequence (ξ_i) are ideals in the excellent order. For $i \geqq 1$, put $F_i = O_{\pi(i)}$ where $\pi(i) = \{\xi_j | j \leqq i\}$. Thus $\pi(i)$ is filled up. We want to show that $F_i N/F_{i-1} N$ is isotypical of type $P(\xi_i)$. We argue by induction and assume $F_j N/F_{j-1} N$ has the desired form for $j<i$. Take λ in $P^- \cap W(-\xi_i)$ and take $w \in W$ minimal so that $-\xi_i = w\lambda$. Put $M = N/F_{i-1} N$. The socle of $F_i M$, if not zero, has weight $-w\lambda$. Suppose $v \in N_{-w\lambda}$ is such that its image $\bar{v}$ in M is a non-zero element of the socle of $F_i M$. We seek a copy of $P(-w\lambda)$ in M that contains $\bar{v}$. This will take many steps. First let $f\colon P(w\lambda)^* \to N$ map $e_{-w\lambda}$ to v as in 2.4. If μ is a weight of the image of f and $(\mu, \mu) = (\lambda, \lambda)$, then $-\mu$ is a weight of $P(w\lambda)$ so that there is $z \in W$ with $-\mu = z\lambda$, $z \leqq w$. (Proof of 2.6.) Thus $-w\lambda \leqq -z\lambda = \mu$, which together with $\mu \in \pi(i)$ implies $\mu = -w\lambda$. This means that v is a longest weight vector. Let $\phi\colon Q(w\lambda)^* \to N$ be the corresponding map (Prop. 2.7). Let Ω be the set of $z \in W$ with $P(z\lambda)_{w\lambda} = 0$. Put $S_1 = X_\Omega$, $S_2 = X_\Omega \cup X_w$, $S_3 = G/B$. In the exact sequence $0 \to K_{32} \to K_{31} \to K_{21} \to 0$ of the proof of 2.2 we now have $K_{21} = Q(w\lambda)$, because $z<w$ implies $z \in \Omega$ (2.6). As N is injective, we may extend $\phi\colon K_{21}^* = Q(w\lambda)^* \to N$ to a map $K_{31}^* \to N$. That in turn yields a map ψ from the Weyl module $H^0(\lambda)^* = \Gamma(G/B, \mathscr{L}(\lambda))^*$ to N which vanishes on $H_z(\lambda)^*$ with $z \in \Omega$. Note the weights $-z\lambda$ with $z \in \Omega$ occur with multiplicity one in both $H^0(\lambda)^*$ and $H_z(\lambda)^*$, so they do not occur in the image of ψ. We get

2.14. **Lemma.** *The image of ψ lies in $F_i N$. If $z \in W$ is such that $-z\lambda$ is a weight of image (ψ), then $-z\lambda \leqq -w\lambda$.*

Proof. Let μ be a weight of the image. If $(\mu, \mu) < (\lambda, \lambda)$ then $\mu \in \pi(i)$. If $(\mu, \mu) = (\lambda, \lambda)$ then $\mu = -z\lambda$ for some $z \in W$, $z \notin \Omega$. Then $P(z\lambda)_{w\lambda} \neq 0$, so $w\lambda \leqq z\lambda$ and $-z\lambda \leqq -w\lambda$.

2.15. The G radical of the Weyl module $H^0(\lambda)^*$ is mapped by ψ to a submodule of $F_{i-1} N$ because its weights are strictly shorter than λ. The composite of ψ with the projection $N \to M$ therefore factors through the irreducible G module $L(-\lambda)$ of highest weight $-\lambda$. We now have a map $L(-\lambda) \to F_i M$ whose image still contains $\bar{v}$. We wish to extend it to $P(-\lambda)$ and therefore now claim that $\mathrm{Ext}_B^1(P(-\lambda)/L(-\lambda), M)$ vanishes. In fact we claim that $\mathrm{Ext}_B^j(R, M)$ vanishes for every $j \geqq 0$ and every finite dimensional G module R all of whose weights are strictly shorter than λ. To prove this, it suffices – as is well known – to take for R a Weyl module $H^0(\mu)^*$ with $\mu \in P^-$, $(\mu, \mu) < (\lambda, \lambda)$. (Recall that one uses long exact sequences for Ext and argues by induction on the length of the longest weight in R and on the dimension of R). Now to see that $\mathrm{Ext}_B^j(H^0(\mu)^*, M)$ vanishes, one first observes that $F_{i-1} M = 0$ settles the case $j = 0$. As N is injective, it remains to show that $\mathrm{Ext}_B^j(H^0(\mu)^*, F_{i-1} N) = 0$ for $j>0$, ($j>1$ would do.) Now $F_{i-1} N$ has an excellent filtration, by induction hypothesis, so it suffices to make $\mathrm{Ext}_B^j(H^0(\mu)^*, H_z(v))$ vanish for $v \in P^-$, $z \in W$, $j>0$. But $\mathrm{Ext}_B^j(H^0(\mu)^*, v) = H^j(B, H^0(\mu) \otimes v) = H^j(G, H^0(\mu) \otimes H^0(v))$ vanishes for $j>0$, $\mu, v \in P^-$ by Cline, Parshall, Scott [J, II 4.13], and the result follows by induction on $l(z)$. (Apply [J, Prop. I 4.5], using the acyclicity of the $P(zv)$ for the induction functors H_y, which was discussed in 2.11.)

2.16. Now that we have the vanishing of $\mathrm{Ext}_B^1(P(-\lambda)/L(-\lambda), M)$, we use it to extend the map $L(-\lambda) \to F_i M$ to a map $P(-\lambda) \to M$. As the weights of

$P(-\lambda)/L(-\lambda)$ are strictly shorter than λ, we actually still land inside $F_i M$. But what we really want is a map $P(-w\lambda)\to F_i M$, so we must show that $P(-\lambda)\to M$ factors through $P(-w\lambda)$. Consider the image of $K=\ker(P(-\lambda)\to P(-w\lambda))$ in $F_i M$. If this image is not trivial, consider a weight μ of its socle. As $F_{i-1}M=0$, μ must be equal to $-w\lambda$ and thus $K_{w\lambda}\neq 0$, contradicting the definition of K. This means that K must have image zero and that $P(-\lambda)\to F_i M$ factors through $P(-w\lambda)$.

2.17. For every $\bar{v}$ in the socle of $F_i M$ (if $F_i M\neq 0$), we thus have a map $P(-w\lambda)\to F_i M$ with $\bar{v}$ in the image. Moreover $P(-w\lambda)\to F_i M$ is injective (if $\bar{v}\neq 0$) because it is injective on socles. In the category $\mathscr{C}[R]$ of Corollary 2.5 we see that $F_i M$ contains an injective hull of its socle (which is of weight $-w\lambda$) so that $F_i M$ is isotypical of type $P(-w\lambda)$. Theorem 2.12 follows.

2.18. *Remark.* What did we use about N? We did not use the full force of injectivity, but only that $\mathrm{Ext}^1_B(K^*_{32}, N)=0$ in 2.13 and that $\mathrm{Ext}^1_B(P(-\lambda)/L(-\lambda), N/F_{i-1}N)$ vanishes in 2.15. Before we can weaken these conditions further, we must first exploit the theorem.

2.19. *Proof of Theorem 1.6.* (i) Let ξ_i, F_i be as in 2.13 and consider the excellent filtration $\{F_j(I(\xi_i))\}$ of an injective hull of ξ_i. As the socle of $I(\xi_i)$ is ξ_i [J, I 3.17], we must have $F_j(I(\xi_i))=0$ for $j<i$ and $F_i(I(\xi_i))=P(\xi_i)$. (This could serve as the definition of $P(\xi_i)$ and follows from 2.5 and the fact that F_i sends $I(\xi_i)$ to the injective hull in $\mathscr{C}[\pi(i)]$ of ξ_i.) We still have to see that $F_j(I(\xi_i))/F_{j-1}(I(\xi_i))$ is finite dimensional for each j. But that is an obvious consequence of the fact that $I(\xi_i)\cong\mathrm{ind}_T^B(\xi_i)$ has finite dimensional weight spaces [J, II 4.8].

(ii) Let λ be a weight and let $I(\lambda)$ be an injective hull of λ. Put $R=(\lambda,\lambda)$ and $\pi=\{\mu\mid(\mu,\mu)\leq R\}$. As in part (i) we find $O_\pi(I(\lambda))=P(\lambda)$ and $I(\lambda)/P(\lambda)$ has an excellent filtration with layers that are isotypical of type $P(\nu)$ with $(\nu,\nu)>R$. If we refine the excellent filtration from (i) of $I(\lambda)$ we thus get layers of $P(\lambda)$ as in 2.8 followed by layers of type $Q(\nu)$ with $(\nu,\nu)>R$. The very first layer is $Q(\lambda)$ (because of socles) and after that we get the other $Q(\mu)$ with $P(\lambda)_\mu\neq 0$, $\mu\in W\lambda$, hence with $\mu\geq\lambda$ in antipodal excellent order, cf. 2.6. Finish as in part (i).

2.20. We are also ready to prove Theorem 1.7 (a) (ii).

Theorem. (i) *Let M have an excellent filtration and N a relative Schubert filtration. Then $M\otimes N$ is B acyclic.*

(ii) *If M and N both have excellent filtrations, then $M\otimes N$ is B acyclic.*

Comment. By 2.2, part (ii) is a special case of part (i). Part (ii) was used in [P, Prop. 2.11] and that proposition suggested the more general part (i).

Proof. We show by induction on j, $j\geq 1$, that $H^j(B, M\otimes N)=0$ for M, N as in (i). The case $j=1$ follows from 2.10 by dévissage (and of course a limit argument, cf. 2.1). Assuming the result for $1\leq j\leq n$, where $n\geq 1$, we consider an injective hull $I(\lambda)$ of some λ and use the exact sequence

$$H^n(B,(I(\lambda)/P(\lambda))\otimes N)\to H^{n+1}(B,P(\lambda)\otimes N)\to H^{n+1}(B,I(\lambda)\otimes N)$$

to see that $H^{n+1}(B, P(\lambda)\otimes N)$ vanishes. (Note that $I(\lambda)/P(\lambda)$ has an excellent filtration and that $I(\lambda)\otimes N$ is injective [J, Prop. I 3.10]). The induction step finishes by dévissage. (This reasoning is of course standard for highest weight categories.)

2.21. Next we prove 1.7 (a) (i).

Corollary. *A module with excellent filtration is acyclic for Long.*

Proof. $R^i \operatorname{Long}_\lambda M = \operatorname{Ext}^i_B(Q(-\lambda)^*, M) = H^i(B, Q(-\lambda)\otimes M) = 0$ for $i>0$ and λ any weight, by 2.7 and 2.20.

2.22. Let us prove 1.7 (a) (iii) and thus finish the proof of 1.7 (a). The following theorem actually corresponds with a general fact about highest weight categories. (One must take π to be an ideal in Λ.)

Theorem. *If π is filled up and M has an excellent filtration, then M is acyclic for O_π and $O_\pi M$ has an excellent filtration* (*cf.* [D]).

Proof. Let us prove by induction on i, $i\geqq 1$, that $R^i O_\pi(M)=0$ whenever M has an excellent filtration. As in 2.20 we consider an injective hull $I(\lambda)$ of $P(\lambda)$ for some weight λ. We may choose the sequence $\xi_1, \xi_2, \ldots$ in 2.13 such that π is one of the initial segments and it is then clear that $O_\pi(I(\lambda))\to O_\pi(I(\lambda)/P(\lambda))$ is surjective. That shows $R^1 O_\pi(P(\lambda))=0$ and starts the induction. It finishes as in 2.20. That $O_\pi M$ has an excellent filtration then follows from the fact that O_π sends the isotypical module $F_j M/F_{j-1} M$ to itself or to zero.

2.23. *Exercise.* Let $w, y\in W$ with y simple. Use Lemma 2.11, Theorem 2.12 and the proof of 2.11 to set up a Grothendieck spectral sequence [J, Prop. I 4.1] $H^m_y(H^n_w(M))\Rightarrow H^{m+n}_{y*w}(M)$. Use it to prove by induction on $l(w)$ that a module with excellent filtration is acyclic for H_w. (Andersen proved this, using Leray spectral sequences, in [A], as he explained in a lecture at Durham.) Compare also [P, Prop. 1.4.2].

2.24. We now wish to apply the same reasoning as in the exercise to prove Theorem 1.9 (a) (ii). We start with an analogue of [P, Prop. 2.10].

Proposition. *Let y be a simple reflection and let S_1, S_2, λ be as in* 2.2.

(i) *$H_y(\Gamma(S_i, \mathscr{L}(\lambda)))=\Gamma(y*S_i, \mathscr{L}(\lambda))$, where $y*S_i$ denotes the union (with reduced subscheme structure again) of the X_{y*w} with $X_w\subseteq S_i$.*

(ii) *$H_y(\Gamma(S_2, \mathscr{L}(\lambda)))\to H_y(\Gamma(S_1, \mathscr{L}(\lambda)))$ is surjective.*

Proof (cf. [P]). Part (ii) follows from part (i) by [R, Th. 2]. We prove part (i) by induction on the size of S_i. Thus let S_2 be non-empty and such that $H_y(\Gamma(S_0, \mathscr{L}(\lambda)))=\Gamma(y*S_0, \mathscr{L}(\lambda))$ whenever S_0 is properly contained in S_2. As in 2.2 we may choose S_1 and w such that $S_2=S_1\cup X_w$, $S_1\cap X_w=\partial X_w$. We get an exact sequence

$$0\to\Gamma(S_2, \mathscr{L}(\lambda))\to\Gamma(X_w, \mathscr{L}(\lambda))\oplus\Gamma(S_1, \mathscr{L}(\lambda))\to\Gamma(\partial X_w, \mathscr{L}(\lambda))\to 0.$$

By the induction hypothesis, and [R, Th. 2] again, this yields an exact sequence

$$0 \to H_y(\Gamma(S_2, \mathcal{L}(\lambda))) \to \Gamma(y * X_w, \mathcal{L}(\lambda)) \oplus \Gamma(y * S_1, \mathcal{L}(\lambda)) \to \Gamma(y * \partial X_w, \mathcal{L}(\lambda)) \to 0.$$

As in [P] we have $(y * X_w) \cap (y * S_1) = y * \partial X_w$, scheme theoretically, and (i) follows for S_2.

2.25 As H_y^i vanishes for $i \geqq 2$ by [J, Prop. I 5.12], the proposition shows, when applied to $S_2 = X_w$, $S_1 = \partial X_w$, that $Q(\mu)$ is acyclic for H_y and that $H_y(Q(\mu))$ has a relative Schubert filtration. (μ any weight.) Theorem 1.9 (a) (ii) follows as in exercise 2.23.

§ 3. Cohomological Criteria

3.1. Let us prove case (ii) of Theorem 1.7 (b).

Theorem. *Let M be a B module such that $H^1(B, L \otimes M)$ vanishes for every B module L with relative Schubert filtration. Then M has an excellent filtration.*

Proof. (For another proof see exercise 3.6). Note that Theorem 3.1 is stronger than Theorem 2.12. We now explain how to modify the proof of 2.12 so as to get the present theorem. If $F_{i-1}M$ has an excellent filtration, then $M/F_{i-1}M$ also satisfies the conditions of the theorem, by the long exact sequence and Theorem 2.20. We may therefore assume $F_{i-1}M = 0$ and must then prove that $F_i M$ is isotypical of type $P(\xi_i)$. The distinction between M and N in 2.13 should be ignored now. The module K_{32} of 2.13 has a relative Schubert filtration (2.2) so $\mathrm{Ext}_B^1(K_{32}^*, M) = H^1(B, K_{32} \otimes M)$ vanishes by hypothesis. Remains to show (2.18) that $\mathrm{Ext}_B^1(P(-\lambda)/L(-\lambda), M)$ vanishes, where $\lambda \in P^- \cap W(-\xi_i)$. We claim that in fact both $\mathrm{Ext}_B^0(R, M)$ and $\mathrm{Ext}_B^1(R, M)$ vanish for every G module R (say finite dimensional for simplicity) all of whose weights are strictly shorter than λ. To see this, we first observe as before that the vanishing of $F_{i-1}M$ implies the Ext^0 statement. Next if $\mu \in P^+$ with $(\mu, \mu) < (\lambda, \lambda)$, consider the simple G module $L(\mu)$ of highest weight μ as a submodule of $P(\mu)$ and use the exact sequence

$$\mathrm{Hom}_B((P(\mu)/L(\mu))^*, M) \to \mathrm{Ext}_B^1(L(\mu)^*, M) \to \mathrm{Ext}_B^1(P(\mu)^*, M).$$

The first term vanishes, as does the third, because $P(\mu)$ has a relative Schubert filtration. Finish by dévissage.

3.2. **Corollary** (*Case (ii) of Theorem 1.7 (b)*).

If R^1 Long $M = 0$ then M has an excellent filtration.

Proof. For each λ the group $R^1\,\mathrm{Long}_\lambda M = H^1(B, Q(\lambda) \otimes M)$ vanishes. Therefore the hypothesis of 3.1 is satisfied.

3.3. Now let us do case (iii) of Theorem 1.7 (b), thereby finishing the proof Theorem 1.7.

Theorem. *Let M be a B module such that $R^1 O_\pi(M) = 0$ whenever π is filled up. Then M has an excellent filtration* (*cf.* [D, 2.1 d]).

Proof. As in 3.1 we may assume $F_{i-1}M=0$ and must then show that F_iM is isotypical of type $P(\xi_i)$. Let $I(M)$ be an injective hull of M. Then $0\to F_{i-1}M \to F_{i-1}I(M)\to F_{i-1}(I(M)/M)\to 0$ is exact. If $F_{i-1}I(M)$ is not zero, then its socle is contained in the socle of $I(M)$, hence of M, hence of $F_{i-1}M$, which is zero. So $F_{i-1}I(M)$ vanishes and $F_{i-1}(I(M)/M)$ vanishes. Now suppose that $F_i(I(M)/M)$ is not zero. Its socle is then of weight ξ_i, as is the socle of $F_iI(M)$. Consider the exact sequence $0\to F_iM\to F_iI(M)\to F_i(I(M)/M)\to 0$. Take v in $F_iI(M)$ such that its image $\bar{v}$ in $F_i(I(M)/M)$ is a non-zero element of the socle. In the B module generated by v the socle S has the same weight as $\bar{v}$, so its image contains $\bar{v}$. On the other hand, S is contained in M, hence in F_iM, and therefore has trivial image in $F_i(I(M)/M)$. Contradiction. This means that F_iM equals $F_iI(M)$ which is indeed isotypical of type $P(\xi_i)$.

Remark. The result holds in any highest weight category which shares the following property with the category of B modules. The injective hull of an irreducible has that irreducible only once as a composition factor.

3.4. Corollary 1.8 being obvious, we now turn to Theorem 1.9 (a) (i). But that one is also clear, as it is another case of the "general fact" in 2.22, which is proved like Theorem 2.22. This finishes the proof of 1.9 (a). Of course 1.9 (b) is a somewhat different story. (Compare 1.14.)

3.5. **Theorem.** *(Case (i) of Theorem 1.9 (b)). Let M be a B module such that $H^1(B, M\otimes N)=0$ whenever N has an excellent filtration. Then M has a relative Schubert filtration.*

Proof. Now choose a linear order $\eta_1, \eta_2, \ldots$ of the weights so that the initial segments of (η_i) are ideals for the antipodal excellent order. Put $\pi(i)=\{\eta_j \mid j\leqq i\}$ and $F_i=O_{\pi(i)}$, just like in 2.13. As in 3.1 we may assume $F_{i-1}M=0$ and we must show that F_iM is isotypical of type $Q(\eta_i)$. Put $\lambda=\eta_i$ and $R=(\lambda,\lambda)$. Let $I(M)$ be an injective hull of M. If μ is a weight, then $0\to \mathrm{Hom}_B(P(\mu)^*, M) \to \mathrm{Hom}_B(P(\mu)^*, I(M))\to \mathrm{Hom}_B(P(\mu)^*, I(M)/M)\to 0$ is exact, so that by 2.4 the socle of $I(M)/M$ has no weight $-\mu$ with $(\mu,\mu)<R$. (Compare proof of 3.3.) Suppose $F_i(I(M)/M)$ is not zero and let μ be a weight of its socle. Choose a non-trivial homomorphism $P(-\mu)^*\to I(M)/M$, lift it to a map ϕ: $P(-\mu)^*\to I(M)$ and consider a weight ν in the socle of the image of ϕ. This ν is a weight of the socle of M, so if $\nu\neq\lambda$, then ν does not precede λ in the sequence (η_i). But μ precedes λ and ν is a weight of $P(-\mu)^*$, so ν precedes μ. Thus $\lambda=\mu=\nu$ and the image of ϕ is contained in F_iM. But that contradicts the choice of $P(-\mu)^*\to I(M)/M$. As in 3.3 we have seen that $F_iM=F_iI(M)$, which is isotypical of type $Q(\eta_i)$. (Use 1.6 (ii).)

3.6. *Exercise.* Taking the above proof as a model reprove 3.2 and derive 3.1 from it.

3.7. To finish the proof of Theorem 1.9 we show

Theorem. *Let M be a B module such that $R^1O_\pi(M)=0$ whenever π is round. Then M has a relative Schubert filtration.*

Proof. Again we may assume $F_{i-1}M=0$ and must show that F_iM is isotypical of type $Q(\eta_i)$, where F_i is as in 3.5. Using $\pi=\{\mu \mid (\mu,\mu)<(\eta_i,\eta_i)\}$, which is round, show that $I(M)/M$ has no weights of length strictly shorter than η_i in its socle, and proceed as in 3.5, using 2.5 and surjectivity of $O_{\pi[R]} I(M) \to O_{\pi[R]}(I(M)/M)$ where $\pi[R] = \{\mu \mid (\mu, \mu) \leqq (\eta_i, \eta_i)\}$.

3.8. Theorem 1.11 does not depend on Theorem 2.12 and could therefore have been proved a lot earlier. As the sum in the right hand side of 1.11 may "diverge", let us first elaborate on the intended meaning. We reformulate 1.11 as follows.

Theorem. *Let M be a B module whose weight spaces are finite dimensional and let μ be a weight. Then*

$$\dim M_\mu \leqq \sum_\lambda \dim(\mathrm{Long}_\lambda M) \cdot \dim(P(\lambda)_\mu).$$

If equality holds for all μ, then M has an excellent filtration (and conversely).

Remark. If the right hand side in this theorem diverges, so much the better.

3.9. We need the following lemma for the proof of 3.8.

Lemma. *If π is filled up, then $0 \to \mathrm{Long}\, O_\pi M \to \mathrm{Long}\, M \to \mathrm{Long}(M/O_\pi M) \to 0$ is exact.*

Proof. If $\bar{v}$ is a longest weight vector of weight λ in $M/O_\pi M$, lift it to a weight vector v in M and observe that $\lambda \notin \pi$. Hit v with a map $P(-\lambda)^* \to M$ and consider a weight μ of the image with $(\mu, \mu) = (\lambda, \lambda)$. If $\mu \neq \lambda$ then $\mu \in \pi$ but also $\lambda \leqq \mu$ (use 2.6), which is impossible.

3.10. By the lemma, both sides of the inequality in 3.8 are additive over exact sequences $0 \to O_\pi M \to M \to M/O_\pi M \to 0$. Therefore we may reduce to the case where $\mathrm{Long}\, M = \mathrm{Long}_\lambda M$ for some λ. Then M is an object of the category $\mathscr{C}[R]$ of Corollary 2.5 and Long M is also the socle. The result is now obvious from Corollary 2.5.

3.11. Remains to prove Theorem 1.13, and some conjectures [P]. We let $B \times B$ act on $k[B]$ through the formula $((g, h)f)(x) = f(g^{-1}xh)$, cf. [J, I 3.3]. As $k[B]$ is an injective $1 \times B$ module, we may filter it so that $F_i k[B]/F_{i-1} k[B]$ is isotypical, as a $1 \times B$ module, of type $Q(\eta_i)$, where (η_i) is as in 3.5.

Theorem. *The $F_i k[B]$ are $B \times B$ submodules and $F_i k[B]/F_{i-1} k[B]$ is isomorphic with the exterior tensor product of B modules $P(-\eta_i) \boxtimes Q(\eta_i)$. Compare* [J, Prop. II 4.20].

Proof. The Hochschild-Serre spectral sequence $H^m(B, H^n(1 \times B, M)) \Rightarrow H^{m+n}(B \times B, M)$ for $M = (P(\lambda) \boxtimes P(\mu)) \otimes k[B]$ degenerates and shows that

$$\begin{aligned} H^m(B \times B, M) &= H^m(B, H^0(1 \times B, (P(\lambda) \boxtimes P(\mu)) \otimes k[B])) \\ &= H^m(B, P(\lambda) \otimes P(\mu)) = 0 \quad \text{for } m > 0. \end{aligned}$$

So $k[B]$ has a relative Schubert filtration of $B \times B$ modules. Applying F_i to this filtration collects some of its layers, cf. 2.22, so the $F_i k[B]$ are $B \times B$ modules with relative Schubert filtration. To find the multiplicity of $Q(\lambda) \boxtimes Q(\mu)$ in the finer filtration one computes $H^0(B \times B, (P(-\lambda) \boxtimes P(-\mu)) \otimes k[B])$ and uses

Prop. 2.7, Lemma 2.10. The result is that $Q(\lambda)\boxtimes Q(\mu)$ occurs at most once and that it occurs if and only if $(\lambda, \lambda)=(\mu, \mu)$ and $P(-\mu)_\lambda \neq 0$ (use Prop. 2.4). Thus $F_i k[B]/F_{i-1} k[B]$ has a filtration whose layers are the $Q(\lambda)\boxtimes Q(\eta_i)$ with $(\lambda, \lambda) =(\eta_i, \eta_i)$ and $P(-\eta_i)_\lambda \neq 0$. To find the socle of $F_i k[B]/F_{i-1} k[B]$ if suffices to determine $\mathrm{Long}(F_i k[B]/F_{i-1} k[B])$, because this happens to be one dimensional of weight $-\eta_i \boxtimes \eta_i$. Namely, the only possible longest weight vectors are at weights $\lambda\boxtimes\eta_i$ with $(\lambda, \lambda)=(\eta_i, \eta_i)$ and $P(-\eta_i)_\lambda \neq 0$. Such a longest weight vector gives a non-trivial element of

$$H^0(B\times B, (Q(-\lambda)\boxtimes P(-\eta_i))\otimes(F_i k[B]/F_{i-1} k[B])).$$

Now

$$H^1(B\times B, (Q(-\lambda)\boxtimes P(-\eta_i))\otimes F_{i-1} k[B])$$

vanishes as does

$$H^0(B\times B, (Q(-\lambda)\boxtimes P(-\eta_i))\otimes k[B])=H^0(B, Q(-\lambda)\otimes P(-\eta_i)) \quad \text{if } \lambda \neq -\eta_i.$$

This only leaves room for a longest weight vector at the weight $-\eta_i\boxtimes\eta_i$, which has multiplicity one. We may therefore embed $F_i k[B]/F_{i-1} k[B]$ into the injective hull $I(-\eta_i)\boxtimes I(\eta_i)$ of its socle. In that hull the submodule $P(-\eta_i)\boxtimes Q(\eta_i)$ is the only $B\times B$ submodule with the correct layers (cf. 2.19).

3.12. The Joseph filtration conjecture (C2) of [P, 1.5] may be recast as follows.

Conjecture. Let M be a B module that is acyclic for O_π whenever π is round. Let $\lambda\in P^-$. Then $\lambda\otimes M$ is also acyclic for O_π whenever π is round.

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References

[A] Andersen, H.H.: Schubert varieties and Demazure's character formula. Invent. Math. **79**, 611–618 (1985)

[BALG] Bourbaki, N.: Algèbre, Ch. X. Paris: Masson 1980

[BLIE] Bourbaki, N.: Groupes et algèbres de Lie, Ch. 4–6. Paris: Hermann 1968

[CPS] Cline, E., Parshall, B., Scott, L.: Finite dimensional algebras and highest weight categories. Preprint 1987

[D] Donkin, S.: On Schur algebras and related algebras I. J. Algebra **104**, 310–328 (1986)

[De] Deodhar, V.V.: Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. Invent. Math. **39**, 187–198 (1977)

[F] Friedlander, E.: A canonical filtration for certain rational modules. Math. Z. **188**, 433–438 (1985)

[J] Jantzen, J.C.: Representation of algebraic groups. Orlando: Academic Press 1987

[Jo] Joseph, A.: On the Demazure character formula. Ann. Sci. Éc. Norm. Super. **18**, 389–419 (1985)

[P] Polo, P.: Variétés de Schubert et excellentes filtrations. Preprint 1987. To appear in Astérisque

[R] Ramanathan, A.: Schubert varieties are arithmetically Cohen Macaulay. Invent. Math. **80**, 283–294 (1985)

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