VASERSTEIN'S PRE-STABILIZATION THEOREM OVER COMMUTATIVE RINGS

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§1 Summary. We give a new proof of Vaserstein's Pre-stabilization theorem. This theorem describes $GL_n(A) \cap E(A)$ when n is just below the stable range for $GL_m(A)/E_m(A)$. The new proof works only for commutative rings (or ideals in such rings) but it does not need assumptions on Krull dimension, like the old proofs did. All one needs is the relevant stable range condition. The new ideas in the proof come from Vaserstein's recent treatment of the case n = 2. (See preceding paper).

§2 Statement of the result

2.1. We mostly follow the notations and terminology of Vaserstein's paper [3]. Thus let B be a commutative two sided ideal in the ring A. As the notation suggests, one thinks of B as a ring without unit. The reader is advised to take B = A on first reading. We now introduce a subgroup $\tilde{E}_n = \tilde{E}_n(A,B)$ of $GL_n(B)$, $n \geqslant 2$. It is generated by $E_n(A,B)$, $[E_n(A), GL_n(B)]$ and the matrices of the form $(I+XY)(I+YX)^{-1}$ where X and Y are $n \times n$ matrices over B and A respectively such that $I + XY \in GL_n(B)$ and Y is a diagonal matrix of the form $\begin{pmatrix} y & 0 \\ 0 & I \end{pmatrix}$, $y \in A$.

Here and elsewhere I denotes an identity matrix of size n-1, n or n+1. Recall that for $n \geqslant 3$ one knows (cf.[1](2.2),(3.30)) that $E_n(A,B)$ contains $[E_n(A), GL_n(B)]$. Also recall that $I + XY \in GL_n(B)$ implies $I + YX \in GL_n(B)$. ([2]).

<u>THEOREM</u> Let $n \geqslant 2$ and let B be a commutative ideal in A with $sr(B) \leqslant n$. Then $GL_n(B) \cap E(A,B) = \tilde{E}_n(A,B)$.

<u>EXAMPLE</u> Let B = A be commutative with noetherian maximal spectrum of dimension at most n-1. Then $sr(B) \leqslant n$. Now compare with [1],[2],[3].

§3 The proof

3.1. As in §7 of Vaserstein's paper we may assume A is commutative and $sr(A) \leqslant n$. Let $\pi: GL_{n+1}(A) \to GL_{n+1}(A/B)$ and let N be the subgroup of $GL_{n+1}(A/B)$ consisting of matrices $\begin{pmatrix} 1 & 0 & 0 \\ * & I & * \\ * & 0 & 1 \end{pmatrix}$. (They have "N shape").

For technical reasons we will work within $\pi^{-1}(N)$. That is, all our matrices are n+1 by n+1 and lie in $\pi^{-1}(N)$ (or sometimes N). Thus in $\begin{pmatrix} 1 & 0 & 1 \\ z & I & 0 \\ q & 0 & 1 \end{pmatrix}$ the I denotes 1_{n-1} and in $\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}$ it denotes 1_n.
If B = A, then $\pi^{-1}(N)$ is simply $GL_{n+1}(A)$. Throughout $n \geqslant 2$.

3.2. If $a,s \in GL_n(B)$, we let $M(a,s)$ denote the set of $g \in \pi^{-1}(N)$ that can be written as $\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s \end{pmatrix}$. Here our convention 3.1 dictates, for instance, that the first * is a column of length n whose top coordinate is in B.

3.3. LEMMA. If $M(a,s)$ intersects $M(a',s')$, then $(a')^{-1}as(s')^{-1} \in \tilde{E}_n$ $(a,a',s,s' \in GL_n(B))$.

PROOF. Multiplying $g \in M(a,s) \cap M(a',s')$ from left and right by suitable factors, we reduce to the case $a' = s' = I$,
$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s \end{pmatrix} = \begin{pmatrix} I & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ q & 0 & 1 \end{pmatrix}$. Dividing from the left by $\begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and reorganizing, we further reduce to the case $z = 0$.

Here we assumed that the reader knows how to multiply or conjugate matrices in upper or lower triangular block form, and is willing to apply this frequently. For instance, $\begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} I & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I & * \\ 0 & 1 \end{pmatrix}$.

Also, we changed the values of a and b and used that $\tilde{E}_n$ contains $[E_n(A), GL_n(B)]$. Such details will be left implicit. Look at $\pi(g)$ and you see b has its entries in B. We may now apply lemma 2.14 of [1] to write $g = \begin{pmatrix} \bar{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ q & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s_1 \end{pmatrix}$ with $as_1 \in \tilde{E}_n$, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix} = \begin{pmatrix} \bar{a} & 0 \\ * & 1 \end{pmatrix}$.

We have to show $as \in \tilde{E}_n$. Now $\begin{pmatrix} \bar{a} & 0 \\ 0 & 1 \end{pmatrix}^{-1} g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ q & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s \end{pmatrix}$.

Comparing first columns we see $s = s_1$, so that $as \in \tilde{E}_n$ indeed.

3.4. LEMMA. Let $\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix} \in \pi^{-1}(N)$, $a,s \in GL_n(B)$, $g \in M(a,s)$.

Let ${}^T(0,x_1,\ldots,x_n)$ be the last column of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$.

Then for $y \in Ax_1 + \ldots + Ax_{n-1}$ there are $a' \in a\tilde{E}_n$, $s' \in \tilde{E}_n s$, with $gy^{n+1,1} \in M(a',s')$. (Recall y^{ij} denotes the elementary matrix with y in the j-th entry of row i).

PROOF. Write $g = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * & q \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$. We need an element h' of $E_n(A,B)$ with first column ${}^T(1+qyx_n,-x_1q,\ldots,-x_{n-1}q)$ and first row of the form $(1+qyx_n,x_ny^2z_1q,\ldots,x_ny^2z_{n-1}q)$ with $z_i \in A$. To construct it one may first use bare hands or use lemma 3.22 of [1] to get the first column right, next substract suitable multiples of the first column from the other ones. Using h' one easily constructs $h'' \in \tilde{E}_n$ with first column ${}^T(1+qyx_n,-x_1qy^2x_n,\ldots,-x_{n-1}qy^2x_n)$. (We are using that A is commutative). Now $\begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix}\begin{pmatrix} h'' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ * & 1 \end{pmatrix}$ where k has the first column ${}^T(1+qyx_n,x_1y,\ldots,x_{n-1}y)$, so that

$$\left[\begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix}\begin{pmatrix} h'' & 0 \\ 0 & 1 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ * & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

We get $\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix}\begin{pmatrix} h'' & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ * & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$, and the right hand side equals $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & f \end{pmatrix}$ for some $h \in \tilde{E}_n, f \in GL_n(B)$.

As the left hand side is in $M(I,I)$ we must have $f \in \tilde{E}_n$ by lemma 3.3. We get

$$gy^{n+1,1} = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xy & I \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} =$$

$$\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ * & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * & 0 \\ 0 & I & 0 \\ * & * & 1 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & f \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} =$$

$$\begin{pmatrix} a'' & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ * & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * & 0 \\ 0 & I & 0 \\ * & * & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s'' \end{pmatrix} \text{ with } a'' \in a\tilde{E}_n, s'' \in \tilde{E}_n s.$$

Remembering which entries must be in B, one further simplifies to $gy^{n+1,1} = \begin{pmatrix} a' & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s' \end{pmatrix}$ with $a' \in a\,\tilde{E}_n$, $s' \in \tilde{E}_n s$.

3.5. LEMMA. Let $a,a',s,s' \in GL_n(B)$, $g \in M(a,s)$, $y \in A$.

(i) Let further $h \in M(a',s')$. There are $z \in A$, $a_1 \in a\tilde{E}_n, s_1 \in \tilde{E}_n s$, $a_1' \in a'\,\tilde{E}_n, s_1' \in \tilde{E}_n s'$ such that $gz^{n+1,1} \in M(a_1,s_1)$, $h(z-y)^{n+1,1} \in M(a_1',s_1')$.

(ii) If $gy^{n+1,1} \in M(a',s')$, there are $a_1 \in a\tilde{E}_n, s_1 \in \tilde{E}_n s$ with $(a')^{-1}a_1s_1(s')^{-1} \in \tilde{E}_n$.

PROOF.

(i) Let ${}^T(s_1,\dots,s_n)$ be the last column of s and ${}^T(s'_1,\dots,s'_n)$ the last column of s'. Because A is commutative, $(s_1,\dots,s_{n-1},s'_1,\dots,s'_{n-1},s_n s'_n)$ is unimodular. Further $sr(A) \leqslant n \leqslant 2(n-1)$, so that there are $d_1,\dots,d_n,d'_1,\dots,d'_n$ in A such that $d_n = d'_n = 0$ and $(s_1+d_1 s_n s'_n,\dots,s_{n-1}+d_{n-1}s_n s'_n, s'_1+d'_1 s_n s'_n,\dots,s'_{n-1}+d'_{n-1}s_n s'_n)$ is unimodular. Put $c_i = d_i s'_n$, $c'_i = d'_i s_n$, $x_i = s_i+c_i s_n$, $x'_i = s'_i+c'_i s'_n$. Then ${}^T(0,x_1,\dots,x_n)$ is the last column of $\begin{pmatrix}1&0&0\\0&I&c\\0&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&s\end{pmatrix}$ and ${}^T(0,x'_1,\dots,x'_n)$ is the last column of

$\begin{pmatrix}1&0&0\\0&I&c'\\0&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&s'\end{pmatrix}$, where $c = {}^T(c_1,\dots,c_{n-1})$, $c' = {}^T(c'_1,\dots,c'_{n-1})$.

As $(x_1,\dots,x_{n-1},x'_1,\dots,x'_{n-1})$ is unimodular, we may write $y = z + (y-z)$ with $z \in Ax_1 + \dots + Ax_{n-1}$ and $y-z \in Ax'_1 +\dots+Ax'_{n-1}$. With such a choice for z we may apply lemma 3.4 twice to obtain (i).

(ii) Now put $h = gy^{n+1,1}$ and choose $z,a_1,s_1,\ a'_1,s'_1$ as in (i). Because $gz^{n+1,1} = h(z-y)^{n+1,1}$ it follows from lemma 3.3 that $(a'_1)^{-1}a_1 s_1(s'_1)^{-1} \in \widetilde{E}_n$. We get $(a')^{-1}a_1 s_1(s')^{-1} \in \widetilde{E}_n$.

3.6. LEMMA. Let $g \in \pi^{-1}(N)$. There is $y \in A$, $a \in GL_n(B)$, $s \in \widetilde{E}_n$ such that $gy^{n+1,1} \in M(a,s)$.

PROOF. Choose $c \in A$ so that the lower left entry of $g_1 := gc^{n+1,1}$ is congruent to 1 mod B, say equal to 1+b with $b \in B$. If the bottom row of g_1 is $(1+b,x_1,\dots,x_n)$, then $(1+b,x_1,\dots,x_{n-1},-bx_n)$ is unimodular, so we may find $t_0,\dots,t_{n-1} \in A$ such that $(1+b+t_0bx_n, x_1+t_1bx_n,\dots,x_{n-1}+t_{n-1}bx_n)$ is unimodular. Take $u = (c,0) + (t_0b,\dots,t_{n-1}b)$ and observe that $g_2 = g\begin{pmatrix}I&0\\u&1\end{pmatrix}$ has a bottom row of the form $(y_1,\dots,y_{n+1})$ with $(y_1,\dots,y_n)$ unimodular and $y_{n+1} -1 \in B$. Choose $\begin{pmatrix}I&v\\0&1\end{pmatrix} \in \ker\pi$ such that the lower right entry of $g_3 := g_2\begin{pmatrix}I&v\\0&1\end{pmatrix}$ equals 1. Then choose $\begin{pmatrix}I&0\\w&1\end{pmatrix} \in \pi^{-1}(N)$ so that $g_3\begin{pmatrix}I&0\\w&1\end{pmatrix} = \begin{pmatrix}a&*\\0&1\end{pmatrix}\begin{pmatrix}1&0&0\\ *&I&0\\0&0&1\end{pmatrix}$ with $a \in GL_n(B)$.

Summarizing, we have $g = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ * & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} I & -v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 \\ * & 1 \end{pmatrix}$, which may be rewritten as $g = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ -y & 0 & 1 \end{pmatrix}$ with $y \in A$ and $s \in E_n(B) \subset \widetilde{E}_n$. Clearly $gy^{n+1,1} \in M(a,s)$.

3.7. DEFINITION. For $g \in \pi^{-1}(N)$ choose $y \in A$, $a \in GL_n(B)$, $s \in \widetilde{E}_n$, (as in 3.6), so that $gy^{n+1,1} \in M(a,s)$ and put $F(g) = a\widetilde{E}_n$. We claim this defines a map $F : \pi^{-1}(N) \to GL_n(B)/\widetilde{E}_n$. (It is actually a homomorphism, but at this stage we even do not know that $\widetilde{E}_n$ is normal, so that we must view $GL_n(B)/\widetilde{E}_n$ as a set). To prove the claim, suppose we also have $y' \in A$, $a' \in GL_n(B)$, $s' \in \widetilde{E}_n$ with $g(y')^{n+1,1} \in M(a',s')$. We must show that $a\widetilde{E}_n = a'\widetilde{E}_n$. Now part (ii) of 3.5 tells that $gy^{n+1,1}(y'-y)^{n+1,1} \in M(a',s')$ implies the existence of $a_1 \in a\widetilde{E}_n$, $s_1 \in \widetilde{E}_n s$ with $(a')^{-1}a_1s_1(s')^{-1} \in \widetilde{E}_n$. But now s,s_1,s' are in $\widetilde{E}_n$, so that $a'\widetilde{E}_n = a_1\widetilde{E}_n = a\widetilde{E}_n$.

3.8. LEMMA. Let $g \in \pi^{-1}(N)$.

(i) If $h = \begin{pmatrix} 1 & * \\ 0 & I \end{pmatrix} \in GL_{n+1}(B)$, then $F(hg) = F(g)$

(ii) If $k = \begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix} \in GL_{n+1}(A)$, then $F(kg) = F(g)$.

PROOF.

(i) is very easy.

(ii) is also very easy if $k = \begin{pmatrix} 1 & 0 & 0 \\ * & I & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Remains the case $k = t^{n+1,1}$ with $t \in A$.

Choose $y \in A$, $a \in GL_n(B)$, $s \in \widetilde{E}_n$ with $kg\, y^{n+1,1} \in M(a,s)$ and choose $w \in A$, $a' \in GL_n(B)$, $s' \in \widetilde{E}_n$ with $gw^{n+1,1} \in M(a',s')$. By 3.5(i) there are $z \in A$, $a_1, a'_1 \in GL_n(B)$, $s_1, s'_1 \in \widetilde{E}_n$ so that $kg\, y^{n+1,1}z^{n+1,1} \in M(a_1,s_1)$, $gw^{n+1,1}(z-(w-y))^{n+1,1} \in M(a'_1,s'_1)$. In other words, we get $kg(y+z)^{n+1,1} \in M(a_1,s_1)$ and $g(y+z)^{n+1,1} \in M(a'_1,s'_1)$.

We have to show that $a_1\widetilde{E}_n = a'_1\widetilde{E}_n$, while we know

$$kg(y+z)^{n+1,1} = \begin{pmatrix} a_1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s_1 \end{pmatrix} = k\begin{pmatrix} a'_1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & s'_1 \end{pmatrix}.$$

Multiplying from the left by $\begin{pmatrix}1&0&0\\ *&I&0\\ 0&0&1\end{pmatrix}$ and from the right by $\begin{pmatrix}1&0&0\\ 0&I&*\\ 0&0&1\end{pmatrix}$ and changing notation, we reduce to the situation

$$\begin{pmatrix}a_1&*\\ 0&1\end{pmatrix}\begin{pmatrix}1&0\\ *&I\end{pmatrix}\begin{pmatrix}1&*\\ 0&s_1\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&I&0\\ t&0&1\end{pmatrix}\begin{pmatrix}a'_1&b\\ 0&1\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&I&0\\ *&0&1\end{pmatrix}\begin{pmatrix}1&*\\ 0&s'_1\end{pmatrix}$$

where b has coefficients in B (and a_1, a'_1 are again in $GL_n(B)$, s_1, s'_1, are in $\tilde{E}_n$).

All factors are now in $\pi^{-1}\begin{pmatrix}1&0&0\\ 0&I&0\\ *&0&1\end{pmatrix}$, so that we may argue as in [3], proof of lemma 5: Take transposes, conjugate by the extraneous matrix $\begin{pmatrix}0&0&1\\ 0&1&0\\ 1&0&0\end{pmatrix}\in GL_{n+1}(\mathbb{Z})$ and change notation again. This translates our problem into the following one: Prove $\tilde{E}_n a = \tilde{E}_n a'$ when $a, a' \in GL_n(B), s, s' \in \tilde{E}_n$ and

$$\begin{pmatrix}s&*\\ 0&1\end{pmatrix}\begin{pmatrix}I&0\\ *&1\end{pmatrix}\begin{pmatrix}1&*\\ 0&a\end{pmatrix}=\begin{pmatrix}s'&*\\ 0&1\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&I&0\\ *&0&1\end{pmatrix}\begin{pmatrix}1&*\\ 0&a'\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&I&0\\ *&0&1\end{pmatrix}.$$

But 3.5(ii) does just that: It provides $s_1 \in s'\tilde{E}_n = \tilde{E}_n, a_1 \in \tilde{E}_n a'$ such that $s^{-1}s_1a_1a^{-1} \in \tilde{E}_n$, hence $\tilde{E}_n a = \tilde{E}_n a_1 = \tilde{E}_n a'$.

3.9. LEMMA. Let $g \in \pi^{-1}(N)$, $p \in E_{n+1}(A,B)$. Then $F(pg) = F(g)$.

PROOF. The subgroup of $\pi^{-1}(N)$ consisting of the x with $F(xg) = F(g)$ contains the h of 3.8(i) and the k of 3.8(ii). Therefore it contains $E_{n+1}(A,B)$, by [1] lemma 2.2 (take transposes).

3.10 PROOF OF THE THEOREM. We now view $GL_m(B)$ as a subgroup of $GL(B)$ in the usual way, $m \geq 1$. By stability for K_1([2]) we have $E(A,B) \cap GL_n(B) = E_{n+1}(A,B) \cap GL_n(B)$. Further $\tilde{E}_n$ is contained in $E(A,B)$ by [2]. Let $p \in E_{n+1}(A,B) \cap GL_n(B)$. Then $\tilde{E}_n = F(I) = F(p) = p\tilde{E}_n$ by 3.9, so that $p \in \tilde{E}_n$.

REFERENCES

1. W. van der Kallen, A group structure on certain orbit sets of unimodular rows, J. Algebra 82 (1983), 363-397.

2. L. Vaserstein, On the stabilization of the general linear group over a ring, Mat. Sb.79 No.3 (1969), 405-424. (Math. USSR Sb. 8, No 3 (1969), 383-400).

3. L. Vaserstein, Computation of K_1 via Mennicke symbols, this journal.