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THE RELATIVE K_2 OF TRUNCATED POLYNOMIAL RINGS

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1. Introduction

Let R be a regular ring essentially of finite type over a field of positive characteristic. For $q \gg n > 1$ we compute the image of $K_3(R[t]/(t^n))$ in $K_2(R[t]/(t^q), (t^n \bmod t^q))$ under the boundary map ∂ in the long exact K-theory sequence associated with the ideal $(t^n \bmod t^q)$ in $R[t]/(t^q)$. This computation extends earlier work of the second author and confirms his conjectures (see [8]). The main result is obtained via convenient presentations for related K_2-groups.

2. A presentation with few generators

2.1. In this section we give a presentation for the relative K_2 of a rather special type of radical ideal. This type has universal properties that make it relevant in later sections. The treatment is more general than is necessary for the rest of this paper.

2.2. Let $\mathbf{k}$ be a perfect field of characteristic $p>0$. Let r and s be integers with $1 \le r \le s$, and let I be a proper ideal in the polynomial ring $\mathbf{k}[t_1, \dots, t_s]$ with the following properties:

(i) I is generated by monomials that lie in the subring $\mathbf{k}[t_1, \dots, t_r]$.
(ii) For each j with $1 \le j \le r$ some power of t_j is in I.

In later sections we will only need $r=1$, $s=2$. Put

$$A = \mathbf{k}[t_1, \dots, t_s]/I.$$

We will abuse notation and write the image of t_i in A also as t_i; more generally we often do not make any notational distinction between an element and its residue class. We call an element of A a monomial if it is the image of a monomial.

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2.3. let M be the nilradical of A. Observe: $M=(t_1,\dots,t_r)$ and $A/M=\mathbf{k}[t_{r+1},\dots,t_s]$. It follows that $K_2(A,(t_1,\dots,t_s))=K_2(A,M)$. One has a presentation for $K_2(A,M)$ in terms of *Dennis–Stein symbols:*

generators: $\langle a,b\rangle$, one for every pair $(a,b)\in A\times M\cup M\times A$;

relations: $\langle a,b\rangle=-\langle b,a\rangle$, $\langle a,b\rangle+\langle c,b\rangle=\langle a+c-abc,b\rangle$,

$\langle a,bc\rangle=\langle ab,c\rangle+\langle ac,b\rangle$ for $(a,b,c)\in A\times M\times A\cup M\times A\times M$.

(see [8] for a more detailed discussion).

We shall give a presentation with fewer generators and fewer relations. With this more efficient presentation the word problem becomes easy.

2.4. Theorem. *The relative K-group $K_2(A,M)$ has a presentation as an abelian group with*

generators: $\langle f,t_i\rangle$ where $1\le i\le s$ and $(f,t_i)\in A\times M\cup M\times A$;

relations: (1) $\langle f,t_i\rangle+\langle g,t_i\rangle=\langle f+g-fgt_i,t_i\rangle$ *if* $t_if, t_ig\in M$.

(2) *If* $t^\alpha=t_1^{\alpha_1}\cdot\ldots\cdot t_s^{\alpha_s}\in M$, $\alpha_j\ge 0$ *for every* j, *and* $f(X)\in\mathbf{k}[X]$, *then:*

$$\sum \alpha_i\langle f(t^\alpha)t_1^{\alpha_1}\cdot\ldots\cdot t_i^{\alpha_i-1}\cdot\ldots\cdot t_s^{\alpha_s},t_i\rangle=0$$

where the summation is taken over all i with $\alpha_i\ge 1$.

2.5. In order to formulate some corollaries we introduce some more notation. Let $\mathbb{Z}_+$ be the set of non-negative integers. Let $\varepsilon^i=(0,\dots,0,1,0,\dots,0)$ be the ith basis vector in $\mathbb{Z}_+^s$. For $\alpha\in\mathbb{Z}_+^s$ one writes $t^\alpha=t_1^{\alpha_1}\cdot\ldots\cdot t_s^{\alpha_s}$; so: $t^{\varepsilon^i}=t_i$. Put

$$\Delta=\{\alpha\in\mathbb{Z}_+^s\mid t^\alpha\in I\},$$

$$\Lambda=\{(\alpha,i)\in\mathbb{Z}_+^s\times\{1,\dots,s\}\mid \alpha_i\ge 1 \text{ and } t^\alpha\in M\}.$$

Note that, if δ is in Δ, then $\delta+\varepsilon^i$ is also in Δ for $i=1,\dots,s$.
For $(\alpha,i)\in\Lambda$ set:

$$[\alpha,i]=\min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^i\in\Delta\},$$

$$\mathbf{w}(\alpha,i)=\min\{w\in\mathbb{Z}_+\mid p^w\ge[\alpha,i]\}.$$

Observe that $[\alpha,i]\le[\alpha,j]+1$ if both (α,i) and (α,j) are in Λ.

If $\gcd(p,\alpha_1,\dots,\alpha_s)=1$, let

$$[\alpha]=\max\{[\alpha,i]\mid i \text{ such that } \alpha_i\not\equiv 0 \bmod p\}.$$

Put

$$\Lambda^{00}=\{(\alpha,i)\in\Lambda\mid \gcd(\alpha_1,\dots,\alpha_s)=1 \text{ and } i\ne\min\{j\mid\alpha_j\not\equiv 0 \bmod p,\ [\alpha,j]=[\alpha]\}\},$$

$$\Lambda^0=\{(m\alpha,i)\in\Lambda\mid\gcd(m,p)=1 \text{ and } (\alpha,i)\in\Lambda^{00}\}.$$

For $f(X)\in\mathbf{k}[X]$ and $(\alpha,i)\in\Lambda$ put

$$\Gamma_{\alpha,i}(1-Xf(X))=\langle f(t^\alpha)t^{\alpha-\varepsilon^i},t_i\rangle$$

For $g(t_1,\dots,t_s)=t_ih(t_1,\dots,t_s)\in\sqrt{I}=\text{radical }(t_1,\dots,t_r)$ of I, put

$$\Gamma_i(1-g(t_1,\dots,t_s))=\langle h(t_1,\dots,t_s),t_i\rangle.$$

Thus $\Gamma_{\alpha,i}(1-Xf(X))$ equals $\Gamma_i(1-t^\alpha f(t^\alpha))$ for $f(X)\in\mathbf{k}[X]$.

If $1\le i\le r$, hence $t_i\in\sqrt{I}$, Γ_i induces a homomorphism from the multiplicative group $(1+t_i\mathbf{k}[t_1,\dots,t_s]/t_iI)^*$ to $K_2(A,M)$. If $r<i\le s$, then Γ_i induces a homomorphism from $(1+t_i\sqrt{I}/t_iI)^*$ to $K_2(A,M)$. And, if $(\alpha,i)\in\Lambda$, then $\Gamma_{\alpha,i}$ induces a homomorphism from $(1+X\mathbf{k}[X]/(X^{[\alpha,i]}))^*$ to $K_2(A,M)$.

2.6. Corollary. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A,M)\simeq\bigoplus_{(\alpha,i)\in\Lambda^0}(1+X\mathbf{k}[X]/(X^{[\alpha,i]}))^*.$$

2.7. Corollary. *Let $\mathscr{B}$ be a basis of $\mathbf{k}$ as a vector space over $\mathbb{F}_p$. Then $K_2(A,M)$ has a presentation, as an abelian group, with*

generators: $\langle bt^{\alpha-\varepsilon^i},t_i\rangle$ where $b\in\mathscr{B}$, $(\alpha,i)\in\Lambda^0$;

relations: $p^{w(\alpha,i)}\langle bt^{\alpha-\varepsilon^i},t_i\rangle=0$.

2.8. *To prove these results* we start with some observations (see also the first section of [8]). As $\mathbf{k}^*$ is p-divisible and $(1+M)^*$ is a p-group, $\{\mathbf{k}^*,1+M\}$ vanishes in $K_2(A,M)\subset K_2(A)$. In particular, $\langle a,f\rangle=\{a,1-af\}=0$ for $a\in\mathbf{k}^*,f\in M$. Using this it is easy to see that $K_2(A,M)$ is generated by elements $\langle f,t_i\rangle$ with $t_if\in M$. In other words, the images of the Γ_i generate $K_2(A,M)$. For $1\le i\le s$ the image of Γ_i is a p-group generated by elements $\langle at^{\alpha-\varepsilon^i},t_i\rangle$ with $a\in\mathbf{k}$, $(\alpha,i)\in\Lambda$. Thus $K_2(A,M)$ is a p-group and we may view it as a module over $\mathbb{Z}_{(p)}=\{mn^{-1}\in\mathbb{Q}\mid\gcd(p,n)=1\}$.

Let $\mathscr{G}_1$ denote the group for which the presentation in Theorem 2.4 is valid. We want to obtain a map $\mathscr{G}_1\to K_2(A,M)$. Therefore we need to check that the relations (1) and (2) hold in $K_2(A,M)$. Relations (1) are known. Incidentally, they also express that Γ_i induces a homomorphism. To prove relations (2), first reduce by means of relations (1) to the case where $Xf(X)$ is a monomial, but not a pth power. Then write $Xf(X)=aX^m$ with $a\in\mathbf{k}$ and $\gcd(p,m)=1$. We have $\sum\alpha_i\langle at^{m\alpha-\varepsilon^i},t_i\rangle=m^{-1}\langle a,t^{m\alpha}\rangle=0$ and this proves relations (2). Note that relations (2) may be written as $\sum_i\alpha_i\Gamma_{\alpha,i}=0$. Thus, if $\alpha_j\not\equiv0 \bmod p$, such a relation tells how to express the image of $\Gamma_{\alpha,j}$ in terms of the $\Gamma_{\alpha,i}$ with $i\ne j$. Together with $p\Gamma_{\alpha,i}=\Gamma_{p\alpha,i}$ this explains how the generators $\langle at^{\alpha-\varepsilon^i},t_i\rangle$ with $(\alpha,i)\in\Lambda$ but $(\alpha,i)\notin\Lambda^0$ can be eliminated. Also the generators of Corollary 2.7 are seen to correspond to a generating set for $K_2(A,M)$ and we have a surjective homomorphism $\mathscr{G}_3\to K_2(A,M)$, where $\mathscr{G}_3$ is the group for which the presentation in 2.7 is valid. If $\mathscr{G}_2$ denotes the right-hand side of 2.6, then we also have a map $\mathscr{G}_2\to K_2(A,M)$, induced by the $\Gamma_{\alpha,i}$.

We have to show that the surjective maps $\mathscr{G}_i\to K_2(A,M)$ are injective. We will give a proof of the injectivity for $\mathscr{G}_3\to K_2(A,M)$. The other two will then follow, because $\mathscr{G}_3\to K_2(A,M)$ factors through $\mathscr{G}_i\to K_2(A,M)$ with $\mathscr{G}_3\to\mathscr{G}_i$ surjective for $i=1,2$. To test injectivity we produce maps from $K_2(A,M)$ to computable targets.

2.9. First assume $I=(t_1,\dots,t_r)^N$ for some $N>1$. Thus Δ is equal to $\{\alpha\in\mathbb{Z}_+^s \mid \alpha_1+\dots+\alpha_r\geq N\}$. Put $C_j=\mathbf{k}[x_1,\dots,x_j,x_1^{-1},\dots,x_j^{-1}][y]/(y^N)$ for $j=0,\dots,s$. Now fix $(\alpha,i)\in\Delta^{00}$. Let $\mathbf{l}=\min\{j\mid \alpha_j\not\equiv 0 \bmod p \text{ and } [\alpha,j]=[\alpha]\}$. Recall that $(\alpha,\mathbf{l})\notin\Delta^{00}$, so that the corresponding generators $\langle at^{m\alpha-\varepsilon^{\mathbf{l}}},t_{\mathbf{l}}\rangle$ are those we have chosen to eliminate. We map $K_2(A,M)$ via $K_2(A)$ to $K_2(C_s)$ by means of the substitution

$$(*)\qquad t_j\mapsto yx_j^{\alpha_j}\quad\text{for } j\neq\mathbf{l},\qquad t_{\mathbf{l}}\mapsto yx_{\mathbf{l}}^{\alpha_{\mathbf{l}}}(x_1^{-\alpha_1}\cdot\ldots\cdot x_s^{-\alpha_s}).$$

Observe that the factors $x_{\mathbf{l}}$ cancel, so that the variable $x_{\mathbf{l}}$ is redundant. By the fundamental theorem we may decompose $K_2(C_j)$, for $1\leq j\leq s$, as

$$K_2(C_{j-1})\oplus N^+K_2(C_{j-1})\oplus N^-K_2(C_{j-1})\oplus K_1(C_{j-1}).$$

Here the K_1-summand is embedded in $K_2(C_j)$ by the rule $g\mapsto\{g,x_j\}$ and

$$N^+K_2(C_{j-1})=\ker(K_2(C_{j-1}[x_j])\to K_2(C_{j-1}))\quad\text{with } x_j\mapsto 0,$$

$$N^-K_2(C_{j-1})=\ker(K_2(C_{j-1}[x_j^{-1}])\to K_2(C_{j-1}))\text{ with } x_j^{-1}\mapsto 0$$

(see [3, 7]). Thus $K_2(C_s)$ is decomposed in many pieces. The piece we are interested in is the $K_1(C_0)$ summand of the $K_1(C_{i-1})$ summand of the $K_2(C_i)$ summand of $K_2(C_s)$. It consists of the elements $\{g,x_i\}=\langle(1-g)x_i^{-1},x_i\rangle$ with $g\in C_0^*$. Composing the homomorphism $K_2(A,M)\to K_2(C_s)$, induced by $(*)$, with the projection onto the summand $K_1(C_0)$ we get a homomorphism

$$\varphi_{\alpha,i}:K_2(A,M)\to K_1(C_0)=\mathbf{k}^*\times(1+y\mathbf{k}[y]/(y^N))^*$$

One checks that $\varphi_{\alpha,i}$ annihilates $\langle bt^{\beta-\varepsilon^j},t_j\rangle$ for $b\in\mathbf{k}$, $(\beta,j)\in\Delta^0$, unless $j=i$ and $\beta\in\mathbb{Z}\alpha$. In the remaining case one has

$$\varphi_{\alpha,i}\langle bt^{m\alpha-\varepsilon^i},t_i\rangle=(1-by^{m|\alpha|})^{\alpha_i}$$

with $|\alpha|=\alpha_1+\ldots+\alpha_s$. This shows that $\varphi_{\alpha,i}$ detects $\langle bt^{m\alpha-\varepsilon^i},t_i\rangle$ if $N>m|\alpha|$ (recall $\alpha_i\not\equiv 0 \bmod p$). The idea is now to detect a given expression by taking N sufficiently large.

2.10. We return to arbitrary I as in 2.2. We wish to show that the map $\mathscr{G}_3\to K_2(A,M)$ is injective (see 2.8). Suppose it is not and let

$$S=\sum_{(\beta,j)\in\Delta^0}\sum_{b\in\mathbf{k}}h_{\beta,j,b}\langle bt^{\beta-\varepsilon^j},t_j\rangle$$

be a non-zero element in the kernel, with $0\leq h_{\beta,j,b}<p^{w(\beta,j)}$ and all but finitely many $h_{\beta,j,b}$ equal to zero. Choose $(\alpha,i)\in\Delta^{00}$, $n\geq 1$, $a\in\mathbf{k}$, so that $h_{n\alpha,i,a}\neq 0$. Choose $N>n|\alpha|p^{w(n\alpha,i)}$, so that $(t_1,\dots,t_r)^N\subset I$. Put $\tilde{A}=\mathbf{k}[t_1,\dots,t_s]/(t_1,\dots,t_r)^N$ and let $\tilde{M}$ be its nilradical. By 2.9 we have the homomorphism $\varphi_{\alpha,i}:K_2(\tilde{A},\tilde{M})\to K_1(C_0)$ which detects $\langle at^{n\alpha-\varepsilon^i},t_i\rangle$. In fact $\varphi_{\alpha,i}$ also detects the expression $\tilde{S}=\sum h_{\beta,j,b}\langle bt^{\beta-\varepsilon^j},t_j\rangle$ in $K_2(\tilde{A},\tilde{M})$, which maps to the element S of $K_2(A,M)$, that we want to detect. Therefore we want to know what happens to the kernel of $\pi: K_2(\tilde{A}, \tilde{M}) \to K_2(A, M)$ under $\varphi_{\alpha,i}$.

The kernel of π is described by the following lemma.

2.11. Lemma (see [6]). *Let J be an ideal contained in the Jacobson radical of the commutative ring D. Let H be another ideal of D. Then the homomorphism $K_2(D, J) \to K_2(D/H, J+H/H)$ is surjective and its kernel is generated by elements $\langle a, b\rangle$ with $(a, b) \in ((J \cap H) \times D) \cup (J \times H)$.*

2.12. The homomorphism $\varphi_{\alpha,i}: K_2(\tilde{A}, \tilde{M}) \to K_1(C_0)$ induces a test map $\psi_{\alpha,i}: K_2(A, M) \to K_1(C_0)/\varphi_{\alpha,i}$ (ker π). Using the lemma one sees that ker π is generated by the elements $\langle bt^{\beta-\varepsilon^j}, t_j\rangle$ with $b \in \mathbf{k}, \beta-\varepsilon^j \in \Delta$.

2.13. Lemma. $\varphi_{\alpha,i}(\ker \pi) \subset (1+(y^{[\alpha,i]|\alpha|}))^*$.

Proof. We must compute all $\varphi_{\alpha,i}\langle bt^{\beta-\varepsilon^j}, t_j\rangle$ with $\beta-\varepsilon^j \in \Delta, b \in \mathbf{k}$. If β is not a multiple of α, then the computation yields zero. Let $\beta = m\alpha$, $m \geq 1$. First consider the case $j = \mathbf{l}$, with $\mathbf{l}$ as in 2.9. We get

$$\langle bt^{m\alpha-\varepsilon^{\mathbf{l}}}, t_{\mathbf{l}}\rangle = -\alpha_{\mathbf{l}}^{-1} \sum_{q \neq \mathbf{l}} \alpha_q \langle bt^{m\alpha-\varepsilon^q}, t_q\rangle$$

with each term of the right-hand sum also in ker π; indeed, either $[\alpha, q] \leq [\alpha, \mathbf{l}]$, hence $m\alpha - \varepsilon^q \in \Delta$, or $[\alpha, q] > [\alpha, \mathbf{l}]$, in which case $p|\alpha_q$ and the term in question is a multiple of $\langle b^p t^{pm\alpha-\varepsilon^q}, t_q\rangle$, while $pm\alpha - \varepsilon^q \in \Delta$. Therefore we may further assume $j \neq \mathbf{l}$. If $j \neq i$, the computation yields zero again and if $j = i$ one gets $(1-by^{m|\alpha|})^{\alpha_i}$ with $m \geq [\alpha, i]$. □

2.14. As in [8] we denote by $\langle g\rangle$ the class of $1-g$ in $K_1(C_0)$, for $g \in C_0$ with $1-g \in C_0^*$. From 2.9 we obtain $\varphi_{\alpha,i}\tilde{S} = \sum h_{m\alpha,i,b}\alpha_{\mathbf{l}}\langle by^{m|\alpha|}\rangle$, where the summation is over $b \in \mathscr{B}$ and m prime to p. Therefore $\psi_{\alpha,i}S = 0$ implies that the highest p-power $P(h_{m\alpha,i,b})$ that divides $h_{m\alpha,i,b}$ satisfies $m|\alpha|P(h_{m\alpha,i,b}) \geq \min(N, [\alpha, i]|\alpha|)$, whenever $h_{m\alpha,i,b}$ is non-zero. In particular,

$$n|\alpha|P(h_{n\alpha,i,a}) \geq \min(n|\alpha|p^{w(n\alpha,i)}, [\alpha, i]|\alpha|).$$

Now recall $h_{n\alpha,i,a} < p^{w(n\alpha,i)}$. So we must have $nP(h_{n\alpha,i,a}) \geq [\alpha, i]$, and hence $nP(h_{n\alpha,i,a})\alpha - \varepsilon^i \in \Delta$. It follows that $P(h_{n\alpha,i,a}) \geq [n\alpha, i]$, hence $P(h_{n\alpha,i,a}) \geq p^{w(n\alpha,i)} > h_{n\alpha,i,a}$. This is absurd.

We have proved Theorem 2.4 and its two corollaries.

2.15. Remark. The test map $\psi_{\alpha,i}$ of 2.12 is basically just the projection onto the (α, i)-component in 2.6 (replace X by $y^{|\alpha|}$ and multiply by the invertible factor $\alpha_{\mathbf{l}}$).

2.16. In 2.7 we decomposed the summands of 2.6 by choosing a basis $\mathscr{B}$ of $\mathbf{k}$. There is a more functorial decomposition of $(1+X\mathbf{k}[X]/(X^{[\alpha,i]}))^*$, involving the ring of Witt vectors $W(\mathbf{k})$ of $\mathbf{k}$. Namely, recall ([8,(3.1.2)] or [1,I §3]) that $(1+X\mathbf{k}[[X]])^*$ is an infinite product, indexed by the positive integers m prime to p, of copies $W(\mathbf{k})_m$ of $W(\mathbf{k})$. One can show that the kernel of the projection onto $(1+X\mathbf{k}[X]/(X^{[\alpha,i]}))^*$ is the product of the ideals $(p^{\mathrm{w}(m\alpha,i)})$ in the discrete valuation rings $W(k)_m$ (observe that $\mathrm{w}(m\alpha,i)=0$ for $m\geq[\alpha,i]$). Thus $(1+X\mathbf{k}[X]/(X^{[\alpha,i]}))^*$ is isomorphic with

$$\prod_{m \text{ prime to } p} W(\mathbf{k})/(p^{\mathrm{w}(m\alpha,i)}).$$

We get:

2.17. Corollary. $K_2(A,M)\simeq \bigoplus_{(\alpha,i)\in A''} W(\mathbf{k})/(p^{\mathrm{w}(\alpha,i)})$.

2.18. Here are some simple examples, not all new.

(a) If $s=1$, then $K_2(A,M)=0$.

(b) $K_2(\mathbf{k}[t_1,\dots,t_r]/(t_1,\dots,t_r)^2)\simeq K_2(\mathbf{k})\oplus\mathbf{k}^{(r-1)r/2}$.

(c) $K_2(\mathbf{k}[t_1,t_2]/(t_1^2,t_2^2))\simeq K_2(\mathbf{k})\oplus\mathbf{k}$ if $p\neq 2$,
$\simeq K_2(\mathbf{k})\oplus\mathbf{k}^3$ if $p=2$ $(\mathbf{k}^3=\mathbf{k}\oplus\mathbf{k}\oplus\mathbf{k})$.

(d) $K_2(\mathbf{k}[t_1,t_2]/(t_1^3,t_2^3))\simeq K_2(\mathbf{k})\oplus\mathbf{k}^4$ if $p\neq 2,3$,
$\simeq K_2(\mathbf{k})\oplus\mathbf{k}^8$ if $p=3$,
$\simeq K_2(\mathbf{k})\oplus\mathbf{k}^2\oplus W(\mathbf{k})/(p^2)$ if $p=2$.

3. Computation of $K_2(\mathbf{k}[t]/(t^q),(t^n \bmod t^q))$

3.1. From Theorem 2.4 and Lemma 2.11 one may derive a presentation for $K_2(R[t]/(t^q),(t^n \bmod t^q))$, or $K_2(R[t]/(t^q),(t^n))$ for short. We now give some details.

3.2. Theorem. *Let* $\mathbf{k}$ *be a perfect field of characteristic* $p>0$. *Let* $q\geq n\geq 1$. *Then the group* $K_2(\mathbf{k}[t]/(t^q),(t^n))$ *has a presentation as an abelian group, with*
generators: $\langle f(t)t^n,t\rangle$ *with* $f(t)\in\mathbf{k}[t]$;
relations:

(1) $\langle f(t)t^n,t\rangle+\langle g(t)t^n,t\rangle=\langle (f(t)+g(t)-t^{n+1}f(t)g(t))t^n,t\rangle$,

(2) $\langle f(t^{mp^s})t^{mp^s-1},t\rangle=0$ *if* $2\mid s, p\nmid m, mp^{s/2}\geq 2n$,
$\langle f(t^{mp^s})t^{mp^s-1},t\rangle=0$ *if* $2\nmid s, p\nmid m, mp^{(s-1)/2}\geq n$,

(3) $\langle f(t)t^q, t\rangle = 0$

with $f(X), g(X) \in \mathbf{k}[X]$.

3.3. If $q=n$, the theorem is trivial. We henceforth assume $q>n$.

3.4. For m prime to p put

$$\mathrm{v}(m) = \min\{s \mid mp^s > n\},$$

$$\mathrm{w}(m) = \min\{s \mid mp^s > q \text{ or } s \text{ even with } mp^{s/2} \geq 2n \text{ or } s \text{ odd with } mp^{(s-1)/2} \geq n\}.$$

Observe that $\mathrm{w}(m)$ is independent of q if $q \geq \max(pn^2-1, 4n^2-1)$. Let Γ denote the homomorphism from $(1+X^{n+1}\mathbf{k}[X]/(X^{q+1}))^*$ to $K_2(\mathbf{k}[t]/(t^q),(t^n))$ given by $\Gamma(1-X^{n+1}f(X)) = \langle f(t)t^n, t\rangle$. The theorem tells what the kernel of Γ is. If we view $(1+X^{n+1}\mathbf{k}[X]/(X^{q+1}))^*$ as a subquotient of $(1+X\mathbf{k}[[X]])^*$ in the obvious way, then we may also identify $K_2(\mathbf{k}[t]/(t^q),(t^n))$ with a subquotient of $(1+X\mathbf{k}[[X]])^*$, hence of $\prod_m W(\mathbf{k})_m$ (see (2.16)). The subgroup $(1+X^{n+1}\mathbf{k}[[X]])^*$ corresponds with $\prod_m p^{\mathrm{v}(m)} W(\mathbf{k})_m$ and $K_2(\mathbf{k}[t]/(t^q),(t^n))$ with $\prod_m p^{\mathrm{v}(m)} W(\mathbf{k})_m / p^{\mathrm{w}(m)} W(\mathbf{k})_m$.
We get:

3.5. Corollary. *$K_2(\mathbf{k}[t]/(t^q),(t^n))$ is isomorphic with the product, over the positive integers m prime to p, of the groups $W(\mathbf{k})/(p^{\mathrm{w}(m)-\mathrm{v}(m)})$. For $q \geq$* $\max(pn^2-1, 4n^2-1)$ *this is also isomorphic with*

$$(1+t\mathbf{k}[t]/(t^{2n}))^*/\{(1+at^n \bmod t^{2n}) | a \in \mathbf{k}\}.$$

Sketch of proof. To see that $\mathscr{Y} = (1+t\mathbf{k}[t]/(t^{2n}))^*/\{1+at^n \mid a \in \mathbf{k}\}$ is isomorphic with the product of the $W(\mathbf{k})/(p^{\mathrm{w}(m)-\mathrm{v}(m)})$ one shows, under the hypothesis on q, that $\mathrm{w}(m)-\mathrm{v}(m) = \min\{s \mid mp^s = n \text{ or } mp^s \geq 2n\}$ and identifies $\mathscr{Y}$ with a quotient of $(1+t\mathbf{k}[[t]])^*$, hence of $\prod_m W(\mathbf{k})_m$. □

3.6. In 3.5 we found, for q sufficiently large, an isomorphism Δ_n from

$$K_1(\mathbf{k}[t]/(t^{2n}),(t))/\{\langle at^n\rangle \mid a \in \mathbf{k}\} = (1+t\mathbf{k}[t]/(t^{2n}))^*/\{1+at^n \mid a \in \mathbf{k}\}$$

to $K_2(\mathbf{k}[t]/(t^q),(t^n))$. This map Δ_n is the same as the map Δ_n of [5]. So the conjecture of [5, p. 412], stating that Δ_n is an isomorphism, has herewith been proved.

3.7. *Let us prove the theorem.* We have a surjective homomorphism $\varrho: \mathbf{k}[t,u]/(u^q) \twoheadrightarrow \mathbf{k}[t]/(t^q)$ sending t to t, u to t^n. Lemma 2.11 describes the kernel of the surjective map $\pi: K_2(\mathbf{k}[t,u]/(u^q),(u)) \to K_2(\mathbf{k}[t]/(t^q),(t^n))$, induced by ϱ, and Theorem 2.4 describes the source of π. Let $\mathscr{G}_4$ be the group for which the presentation of Theorem 3.2 is valid. First we check that the relations (1), (2), (3) hold in

$K_2(\mathbf{k}[t]/(t^q),(t^n))$. Relations (1) are known and relations (3) are obvious. To prove (2) first use (1) and (3) to reduce to the case where f is a monomial; then apply lemma (1.10) of [8] and 2.8. So there is a homomorphism $\xi: \mathcal{G}_4 \to K_2(\mathbf{k}[t]/(t^q),(t^n))$.

3.8. In order to show that ξ is an isomorphism we construct a surjective homomorphism $\tau: K_2(\mathbf{k}[t,u]/(u^q),(u)) \to \mathcal{G}_4$, such that $\pi = \xi \circ \tau$ and $\ker \pi \subset \ker \tau$. We use the result of 2.6 and define for $(\alpha, i) \in \Lambda^{00}$ the corresponding component of τ, $(1 + X\mathbf{k}[X]/(X^{[\alpha,i]}))^* \to \mathcal{G}_4$, by (write $\alpha = (\mathbf{l}, \mathbf{h})$ for simplicity)

$$(1 + Xf(X)) \mapsto \begin{cases} -(n\mathbf{l} + \mathbf{h} - n)\langle f(t^{n\mathbf{l}+\mathbf{h}})t^{n\mathbf{l}+\mathbf{h}-1}, t\rangle & \text{if } i = 1, \\ \langle f(t^{n\mathbf{l}+\mathbf{h}})t^{n\mathbf{l}+\mathbf{h}-1}, t\rangle & \text{if } i = 2. \end{cases}$$

Observe that this does indeed define a homomorphism.

3.9. Lemma. *Let $i \geq n, mi \geq 2n, a \in \mathbf{k}$. Then $i\langle at^{mi-1}, t\rangle$ vanishes in $\mathcal{G}_4$.*

Proof. It suffices to show that $P(i)\langle at^{mi-1}, t\rangle = \langle a^{P(i)}t^{miP(i)-1}, t\rangle$ vanishes, where, as in 2.14, $P(i)$ denotes the highest power of p dividing i. If $P(m) = p^{2r}$, an even power of p, then our element vanishes because $mip^{-r} \geq 2n$, and if $P(m) = p^{2r+1}$, an odd power of p, then our element vanishes because $mip^{-r-1} \geq n$. □

3.10. Remark. In Theorem 3.2 one may replace the relations (2) by the relations $i\langle at^{mi-1}, t\rangle = 0$ of the lemma $(i \geq n, mi \geq 2n, a \in \mathbf{k})$.

3.11. We check $\pi = \xi \circ \tau$. It suffices to compare the images of the generators of 2.7. A generator of the form $\langle bu^{m\mathbf{l}}t^{m\mathbf{h}-1}, t\rangle$, with $b \in \mathscr{B}$, $(m,p) = 1$ and $((\mathbf{l},\mathbf{h}),2) \in \Lambda^{00}$, goes to $\langle bt^{mn\mathbf{l}+m\mathbf{h}-1}, t\rangle$ both ways. For a generator $\langle bu^{m\mathbf{l}-1}t^{m\mathbf{h}}, u\rangle$, with $b \in \mathscr{B}$, $(m,p) = 1$ and $((\mathbf{l},\mathbf{h}),1) \in \Lambda^{00}$ the desired equality is provided by the following computation in $K_2(\mathbf{k}[t]/(t^q),(t^n))$:

$$\begin{aligned} \langle bt^{mn\mathbf{l}+m\mathbf{h}-n}, t^n\rangle &= -\langle bt^n, t^{mn\mathbf{l}+m\mathbf{h}-n}\rangle \\ &= -(mn\mathbf{l} + m\mathbf{h} - n)\langle bt^{mn\mathbf{l}+m\mathbf{h}-1}, t\rangle \\ &= -(n\mathbf{l} + \mathbf{h} - n)\langle bt^{mn\mathbf{l}+m\mathbf{h}-1}, t\rangle \quad \text{(use 3.9 for } m \geq 2). \end{aligned}$$

3.12. Remains to show that $\ker \tau$ contains $\ker \pi$. The ideal $\ker \varrho$ equals $(t^q, u - t^n)$ and its intersection with (u) equals $(ut^{q-n}, u^2 - ut^n)$. However, it is inconvenient to apply Lemma 2.11 directly. Instead we will use:

Lemma. *The kernel of π is generated by the elements*

$\langle at^{i-1}u^j, t\rangle$	*with* $j \geq 1, i \geq q - nj + 1,$	$a \in \mathbf{k}$,
$\langle at^iu^{j-1}, u\rangle$	*with* $j \geq 1, i \geq q,$	$a \in \mathbf{k}$,
$\langle at^{i-1}u^j, t\rangle - \langle at^{i+n-1}u^{j-1}, t\rangle$	*with* $j \geq 2, i \geq 1,$	$a \in \mathbf{k}$,

$$\langle at^i u^{j-1}, u\rangle - \langle at^{i+n}u^{j-2}, u\rangle \qquad with\ j\geq 2, i\geq 0, \qquad , a\in \mathbf{k}.$$

Proof. On the one hand these elements are clearly in ker π, on the other hand one can use them, exploiting the nilpotence of u, to break down the elements suggested by 2.11. For instance, an element of type $\langle (t^n - u)uf(t,u), g(t,u)\rangle$ is first broken up into pieces with f and g monomial; next into pieces with f monomial and $g(t,u)=t$ or u. Such a piece can be written as a sum of elements listed in the lemma (the third type if $g=t$, the fourth type if $g=u$) because $1-(t^n-u)ufg$ is a product, in $(\mathbf{k}[t,u]/(u^q))^*$, of elements $(1-at^iu^j)(1-at^{i+n}u^{j-1})^{-1}$. □

3.13. Lemma.

$$\tau\langle at^{i-1}u^j, t\rangle = \langle at^{i+jn-1}, t\rangle,$$

$$\tau\langle at^i u^{j-1}, u\rangle = -(i+jn-n)\langle at^{i+jn-1}, t\rangle \quad if\ a\in\mathbf{k},\ i,j\geq 1.$$

Proof. The first formula results directly from the definition of τ if $i=m\mathbf{h}$, $j=m\mathbf{l}$ and $((\mathbf{l},\mathbf{h}),2)\in\Lambda^{00}$. Now suppose $i=m\mathbf{h}$, $j=m\mathbf{l}$ and $((\mathbf{l},\mathbf{h}),1)\in\Lambda^{00}$. Then $(\mathbf{h},p)=1$ and $\langle at^{i-1}u^j, t\rangle = -\mathbf{l}\mathbf{h}^{-1}\langle at^iu^{j-1}, u\rangle$ by (2.4); whence

$$\begin{aligned}\tau\langle at^{i-1}u^j, t\rangle &= -\mathbf{l}\mathbf{h}^{-1}\tau\langle at^iu^{j-1}, u\rangle\\ &= \mathbf{l}\mathbf{h}^{-1}(n\mathbf{l}+\mathbf{h}-n)\langle at^{i+jn-1}, t\rangle\\ &= (1+\mathbf{h}^{-1}(\mathbf{l}-1)(n\mathbf{l}+\mathbf{h}))\langle at^{i+jn-1}, t\rangle\\ &= \langle at^{i+jn-1}, t\rangle, \qquad \text{by 3.9.}\end{aligned}$$

Similar arguments prove the second formula in the lemma. □

3.14. Now we have to check that the elements listed in Lemma 3.12 are in ker τ. For the first three types this is clear. For the fourth type with $i\geq 1$ it is equally trivial. Lemma 3.9 shows that

$$\tau(\langle au^{j-1}, u\rangle - \langle at^nu^{j-2}, u\rangle) = \tau(-\langle at^nu^{j-2}, u\rangle) = n(j-1)\langle at^{jn-1}, t\rangle$$

vanishes for $j\geq 2$ and $a\in\mathbf{k}$.

We have proved Theorem 3.2.

3.15. Examples (for q sufficiently large; see 3.5).

(a) $K_2(\mathbf{k}[t]/(t^q),(t^2)) \simeq \mathbf{k}^2$ if $p\neq 3$,
$\simeq W(\mathbf{k})/(p^2)$ if $p=3$.

In particular, $K_2(\mathbb{F}_3[t]/(t^9),(t^2)) \simeq \mathbb{Z}/(9)$, *with generator* $\langle t^2, t\rangle$.

(b) $K_2(\mathbf{k}[t]/(t^q),(t^3)) \simeq \mathbf{k}^4$ if $p\neq 2, 5$,
$\simeq W(\mathbf{k})/(p^3)\oplus\mathbf{k}$ if $p=2$,
$\simeq W(\mathbf{k})/(p^2)\oplus\mathbf{k}^2$ if $p=5$.

3.16. Via the exact sequence

$$K_3(\mathbf{k}[t]/(t^n),(t)) \xrightarrow{\partial} K_2(\mathbf{k}[t]/(t^q),(t^n)) \longrightarrow K_2(\mathbf{k}[t]/(t^q),(t))=0$$

the computation of $K_2(\mathbf{k}[t]/(t^q),(t^n))$ provides a lower bound for $K_3(\mathbf{k}[t]/(t^n),(t))$. For example one gets a surjection from $K_3(\mathbb{F}_2[t]/(t^3),(t))$ onto $\mathbb{Z}/(8)\oplus\mathbb{Z}/(2)$.

4. The image of ∂

4.1. Let R be a (commutative) $\mathbb{F}_p$-algebra. Let $q\geq n\geq 1$. We are interested in the image of ∂ in the long exact sequence

$$\cdots \to K_3(R[t]/(t^n)) \xrightarrow{\partial} K_2(R[t]/(t^q),(t^n)) \longrightarrow K_2(R[t]/(t^q)) \longrightarrow \cdots .$$

In other words, we are interested in the kernel of $K_2(R[t]/(t^q),(t^n)) \longrightarrow K_2(R[t]/(t^q))$. Or, what amounts to the same thing, we are interested in the image of ∂ in

$$\cdots \longrightarrow K_3(R[t]/(t^n),(t)) \xrightarrow{\partial} K_2(R[t]/(t^q),(t^n)) \longrightarrow K_2(R[t]/(t^q),(t)) \longrightarrow \cdots$$

(The last sequence is a summand of the first. Although the source of ∂ is not the same, the image and the target are.)

In section 3 of [8] a homomorphism Δ_n has been constructed from

$$K_1(R[t]/(t^{2n}),(t))/\{\langle at^n\rangle \mid a\in R\}$$

to the image of ∂ (recall that $\langle at^n\rangle$ denotes the class of the unit $1\text{-}at^n$). The image of Δ_n is the subgroup of $\operatorname{im}\partial$ generated by the elements $\langle a^{p^{r-s}-1}t^{mp^r},a\rangle + m\langle a^{p^r}t^{mp^{r+s}-1},t\rangle$ with $a\in R, m,r,s\in\mathbb{Z}$ such that $0\leq s\leq r, \gcd(m,p)=1, mp^r\geq n, mp^{r+s}>n$ [8, theorem (3.5)(1)]. Conjecture (4.1) of [8] is now a theorem:

4.2. Theorem. *Let R be a domain of characteristic $p>0$. Let n and q be positive integers with $q\geq \max(pn^2-1,4n^2-1)$. Then the homomorphism*

$$\Delta_n: K_1(R[t]/(t^{2n}),(t))/\{\langle at^n\rangle \mid a\in R\} \to K_2(R[t]/(t^q),(t^n))$$

is injective.

Proof. By (4.2) of [8] this follows from the previous section. □

4.3. Theorem. *Let R be a regular ring, essentially of finite type over a field of characteristic $p>0$. Let $q\geq n\geq 1$. Then*

$$\operatorname{im}\Delta_n = \operatorname{im}\partial.$$

In other words, $\operatorname{im}\partial$ *is generated by the elements* $\langle a^{p^{r-s}-1}t^{mp^r},a\rangle + m\langle a^{p^r}t^{mp^{r+s}-1},t\rangle$ *with* $a\in R, 0\leq s\leq r$, $\gcd(m,p)=1, mp^r\geq n, mp^{r+s}\geq n+1$.

Remark. In the proof Δ_n will not be needed. One may simply read im Δ_n as a notation for the subgroup generated by the listed elements.

4.4. Corollary (cf. [1, p. 236, theorem (4.1)]; [8, p. 430]). *Let R be as in Theorem 4.3, $n\geq 1$. Then* $\ker(K_2(R[t]/(t^{n+1}))\to K_2(R[t]/(t^n)))$ *is isomorphic with*

$$\begin{array}{ll} \Omega^1_{R/\mathbb{Z}} & \text{if } n\neq 0,\ -1 \pmod p, \\ \Omega^1_{R/\mathbb{Z}}\oplus R/R^{p^r} & \text{if } n=mp^r-1, \gcd(m,p)=1, r\geq 1, n\geq 2, \\ \Omega^1_{R/\mathbb{Z}}/D_{r,R} & \text{if } n=mp^r, \gcd(m,p)=1, r\geq 1. \end{array}$$

Here $D_{r,R}$ is the subgroup of $\Omega^1_{R/\mathbb{Z}}$ generated by the forms $a^{p^j-1}da$ with $0\leq j<r$ (it is also the kernel of the rth power of the Cartier operator [2]).

If $n=1$ and $p=2$, then there is an exact sequence of $\mathbb{F}_2$ vector spaces

$$0\to R/R^2\to K_2(R[t]/(t^2),(t))\to\Omega^1_{R/\mathbb{Z}}\to 0.$$

Of course, this sequence splits, but it does not split naturally.

Proof of the corollary. For the case $n\geq 2$ and the case $p\geq 3$ see remark 2 following theorem (2.5) of [8]. For $n=1$ and $p=2$ the same argument yields the exactness of

$$R/R^2\to K_2(R[t]/(t^2),(t))\to\Omega^1_{R/\mathbb{Z}}\to 0$$

To see the first map in this sequence is injective, compute its composite with the test map dlog: $K_2(S,(t))\to\Omega^2_{S/\mathbb{Z}}$, $\mathrm{dlog}\langle a,b\rangle=(1-ab)^{-1}\mathrm{d}a\wedge\mathrm{d}b$, $S=R[t]/(t^2)$. Then recall that $R^2=\ker(R\to\Omega^1_{R/\mathbb{Z}})$ (see [2, p. 196]). We leave it to the reader to show that there is no natural splitting i.e. none that is functorial in R. □

4.5. *Let us prove Theorem 4.3.* It has been proved in [8, §2] for the following case: R is smooth of finite type over a perfect field **k** of characteristic p and R can be lifted to a smooth $W(\mathbf{k})$-algebra [8, (2.5)]. First we wish to globalize this result using the sheaf properties proved by Vorst for functors like NK_2 (see [9, §1]).

Thus let R be smooth of finite type over a perfect field **k** of characteristic p. Locally R can be lifted to a smooth $W(\mathbf{k})$-algebra (cf. [4, p. 69]). If S is a **k**-algebra, write ∂^S for the boundary map $K_3(S[t]/(t^n))\xrightarrow{\partial^S}K_2(S[t]/(t^q),(t^n))$ and write Δ_n^S for the corresponding map

$$K_1(S[t]/(t^{2n}),(t))/\{\langle at^n\rangle | a\in S\}\to\mathrm{im}\ \partial^S.$$

Using notations as in [9, p. 35], let $\alpha(t)\in\mathrm{im}\ \partial^R$. We have to show that it lies in $\mathrm{im}\,\Delta_n^R$. Following Vaserstein as in [9], we put

$$I=\{r\in R\,|\,\alpha(tX)-\alpha(tX+trY)\in\mathrm{im}\ \Delta_n^{R[X,Y]}\}.$$

This is an ideal in R. Suppose it is a proper ideal. Choose a maximal ideal M around it and choose $d\in R\setminus M$, so that $R[1/d]$ $(=R[d^{-1}])$ is liftable. Then (the image of) $\alpha(t)$ in im $\partial^{R[1/d]}$ is contained in im $\Delta_n^{R[1/d]}$. Choose a polynomial ring

$B=\mathbb{F}_p[A_1,\dots,A_m,D]$ and a homomorphism $\varphi: B\to R$ with $\varphi(D)=d$, so that in im $\partial^{R[1/d]}$ the element $\alpha(t)$ is the image, under the homomorphism induced by φ, of an element $\beta(t)$ of im $\Delta_n^{B[1/D]}$. Choose an integer f so large that $\beta(tX)-\beta(tX+tD^f y)$ lies in the image of $K_2(B[X,Y][t]/(t^q),(t^n))$ in $K_2(B[1/D][X,Y][t]/(t^q),(t^n))$ (this is possible because $\beta(tX)-\beta(tX+tD^f y)$ is a sum of elements of the type $\langle YF, G\rangle$; one may also use more general results of [9]). The fundamental theorem [3] shows that the map

$$K_2(B[X,Y][t]/(t^q))\to K_2(B[D^{-1}][X,Y][t]/(t^q))$$

is injective. Therefore $\beta(tX)-\beta(tX+tD^f Y)$ lies actually in the image of im $\partial^{B[X,Y]}$, hence of im $\Delta_n^{B[X,Y]}$, because $B[X,Y]$ is smooth over $\mathbb{F}_p$ and liftable. Say $\gamma(t,X,Y)$ in im $\Delta_n^{B[X,Y]}$ has image $\beta(tX)-\beta(tX+tD^f Y)$ in im $\partial^{B[1/D][X,Y]}$. It has the same image in im $\partial^{R[1/d][X,Y]}$ as $\alpha(tX)-\alpha(tX+td^f Y)$. Using lemma (1.4) of [9] one finds $g\ge 1$ so that $\gamma(t,X,D^g Y)-\gamma(t,X,0)$ has the same image in im $\partial^{R[X,Y]}$ as $\alpha(tX)-\alpha(tX+td^{f+g}Y)$. Thus $d^{f+g}\in I\subset M$, contradicting the choice of d. It follows that $I=R$. Consequently, $\alpha(tX)-\alpha(tX+tY)$ is in im $\Delta_n^{R[X,Y]}$. Now substitute $X:=1$ and $Y:=-1$.

We have proved Theorem 4.3 for R smooth of finite type over a perfect field of characteristic p.

4.6. Next let R be a regular local ring, essentially of finite type over a field of characteristic p. Then R is a limit of subrings that are regular and of finite type over $\mathbb{F}_p$ (see, for instance, [10, p. 408]). These subrings are smooth over $\mathbb{F}_p$ (see [5, p. 99]), so that the theorem holds for them by 4.5. It follows that the theorem holds for R. We may use essentially the same arguments as in 4.5 to globalize to the general case of Theorem 4.3 (use R_M instead of $R[d^{-1}]$ or view R_M as a limit of subrings $R[d^{-1}]$).

The proof of Theorem 4.3 is now complete.

4.7. It seems possible that the conclusions of Theorem 4.3 and its corollary hold for any normal ring R of characteristic p. On the other hand, the following exercise shows that some hypothesis on R is needed. It provides an example of a (non-normal) domain R of characteristic 2 for which the map $R/R^2\to \mathrm{K}_2(\mathrm{R}[t]/(t^2),(t))$ is not injective (no such example can exist with R normal, by the proof of Corollary 4.4). It also provides an example of a non-normal domain R of characteristic 3 for which the map $R/R^3\to K_2(R[t]/(t^3),(t))$, sending a to $\langle at^2,t\rangle$, is not injective.

4.8. Exercise. Put

$$A=\mathbb{F}_p[X_1,X_2,X_3,X_4,X_5,X_6,X_1^pX_5^{-p},X_2^pX_5^{-p},X_3^pX_6^{-p},X_4^pX_6^{-p}],$$

$$F=X_1X_2X_5^{-1},\qquad G=X_3X_4X_6^{-1},\qquad R=A[F+G].$$

Then R is a subring of $\mathbb{F}_p(X_1,\dots,X_6)$, but not normal:

$$\mathbb{F}_p[F,G]\cap R=(\mathbb{F}_p[F,G]\cap A)[F+G]$$
$$=\mathbb{F}_p[F^p,F^{p+1},F^{p+2},\dots,G^p,G^{p+1},G^{p+2},\dots][F+G].$$

In $K_2(R[t]/(t^p),(t))$ every element is annihilated by p and for $f\in R[t]$ one has:

$$\langle F^p,tf\rangle=\langle X_1^pX_5^{-p}X_2^p,tf\rangle=\langle X_1^pX_5^{-p},X_2^ptf\rangle=\langle X_5^{-p}X_2^p,X_1^ptf\rangle,$$
$$\langle F^p,tf\rangle=\langle (X_1^pX_5^{-p})X_5^p(X_2^pX_5^{-p}),tf\rangle$$
$$=\langle X_1^pX_5^{-p},X_2^ptf\rangle+\langle X_2^pX_5^{-p},X_1^ptf\rangle,$$

whence $\langle F^p,tf\rangle=0$.

If $p=2$ put $a=F^pG^p$. Then $a\in R\setminus R^p$ and

$$\langle at,t\rangle=\langle (F+G)^p,t\rangle+\langle F^p,t\rangle+\langle G^p,t\rangle=0.$$

If $p=3$ put $a=F^{2p}G^p+F^pG^{2p}$. Then $a\in R\setminus R^p$ and

$$\langle at^2,t\rangle=\langle (F+G)^p,t\rangle+\langle -F^p,t\rangle+\langle -G^p,t\rangle+\langle F^pG^p,t^2\rangle$$
$$+\langle F^{2p},t^2\rangle+\langle G^{2p},t^2\rangle=0.$$

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