Math. Ann. 244, 33–53 (1979)

Mathematische Annalen

Localization of the K-Theory of Polynomial Extensions

Ton Vorst*

Mathematical Institute, State University, Budapest Laan, Utrecht 2506, The Netherlands

Introduction

In this paper all rings will be assumed to be commutative and with identity element. A ring A is said to be K_n-regular if for all r one has $K_n(A) \cong K_n(A[T_1, \ldots, T_r])$. One of our main results is that when A is a k-algebra of essentially finite type over a field k with $\dim A \leqq 1$, A is K_2-regular implies A is regular. Since it is well known that for a ring A with $\dim A = 0$, A is K_1-regular implies A is regular, we may now state the following conjecture.

Conjecture. Let A be a k-algebra of essentially finite type over a field k with $\dim A \leqq n$. Then A is K_{n+1}-regular implies A is regular.

As stated above this conjecture is true for $n = 0, 1$. But for higher n it is an open question. Further it is not clear whether the assumption that A is of essentially finite type over a field is necessary.

For affine curves over a field one can now say for every $n \geqq 0$ what K_n-regularity means in geometric terms using [2], [21] and an extension of the above result. We have

(i) A is K_0-regular if and only if A is seminormal ([2]).

(ii) A is K_1-regular if and only if A is seminormal and for every point x, the points x_i of the normalization above x give separable field extensions $k(x) \subset k(x_i)$ ([21]).

And for $n \geqq 2$

(iii) A is K_n-regular if and only if A is regular (Theorem 3.6).

For the last part we need that K_n-regularity implies K_{n-1}-regularity. This was a question of Bass for $n = 1$. (See [3].) In Sect. 2 we shall show that the answer to the question is positive for all n. The above results suggest that also in higher dimensions K_n-regularity classifies singularities.

One of the main techniques we use to prove the above results is that $NK_n(A) = 0$ for a reduced ring if and only if $NK_n(A_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m}$

---

* Present address: Erasmus Universiteit Rotterdam, Faculteit der Economische Wetenschappen, Burgemeester Oudlaan 59, Rotterdam, Nederland

0025-5831/79/0244/0033/$04.20

of A, extending Theorem 1.1 of [21] to higher K-functors. This is proved in Sect. 1, where we furthermore shall show that the cohomology of Spec(A) with coefficients in the sheaf $\mathcal{N}\mathcal{K}_n$ (i.e. a sheaf associated to the functor NK_n) is trivial. Similar results are proved for the functors C_pK_n (curves on K_n) of Bloch and EK_n (K-theory of endomorphisms) of Grayson.

1. NK_n and Localization

1.1. Notations. Let $F:\mathbf{CRg}\to\mathbf{Ab}$ be a functor. ($\mathbf{CRg}$ is the category of commutative rings and $\mathbf{Ab}$ is the category of abelian groups.) If A is a commutative ring, we denote

$$NF(A)=\mathrm{Ker}(F(A[X])\xrightarrow{X\mapsto 0}F(A)).$$

Now let f be an element of A. Consider the ring homomorphism $\phi_f:A[X]\to A[X]$ with $\phi_f(q(X))=q(fX)$. One easily sees that $F(\phi_f)$ induces a group endomorphism $F(\phi_f):NF(A)\to NF(A)$. This enables us to give $NF(A)$ a $\mathbb{Z}[T]$-module structure if we let T act on $NF(A)$ via the endomorphism $F(\phi_f)$. But once we have this $\mathbb{Z}[T]$-module structure we can localize with respect to powers of T, which gives us a $\mathbb{Z}[T,T^{-1}]$-module $\mathbb{Z}[T,T^{-1}]\otimes_{\mathbb{Z}[T]}NF(A)$ which will be denoted by $NF(A)_{[f]}$.

As usual one can see such a localization as a direct limit as follows. Take $NF(A)^n=NF(A)$ and $\phi_f^n:NF(A)^n\to NF(A)^{n+1}$ the homomorphism $F(\phi_f)$. Then $NF(A)_{[f]}\cong\varinjlim_n(NF(A)^n,\phi_f^n)$ as abelian groups. This description will be useful later.

To a functor $F:\mathbf{CRg}\to\mathbf{Ab}$ and a commutative ring A we can associate a presheaf F of abelian groups on $X=\mathrm{Spec}(A)$ by defining $\mathsf{F}(U)=F(\Gamma(U))$ where U is an open subset of X and $\Gamma(U)$ are the sections of the structure sheaf $\mathcal{O}_X$ on U. (See [13], Chap. I, §1.1.) So for an $f\in A$ we have $\mathsf{F}(D(f))=F(A_f)$. Now let $f_0,\dots,f_r\in A$ be a unimodular row of A. (This means that the ideal of A generated by $f_0,\dots,f_r$ is A itself.) Then $D(f_0),\dots,D(f_r)$ give an open covering of X. So we can form the Čech complex of this covering with coefficients in the presheaf F.

$$0\to F(A)\xrightarrow{d_0}\prod_{0\le i\le r}F(A_{f_i})\xrightarrow{d_1}\prod_{0\le i_0<i_1\le r}F(A_{f_{i_0}f_{i_1}})\xrightarrow{d_2}\dots\xrightarrow{d_r}F(A_{f_0\dots f_r})\to 0$$

with

$$(d_p(\alpha))_{i_0,\dots,i_p}=\sum_{k=0}^{p}(-1)^k(\alpha)_{i_0,\dots,\hat{i}_k,\dots,i_p}$$

where $(d_p(\alpha))_{i_0,\dots,i_p}$ denotes the component of $d_p(\alpha)$ in $F(A_{f_{i_0}\dots f_{i_p}})$, and $(\alpha)_{i_0,\dots,\hat{i}_k,\dots,i_p}$ is the image in $F(A_{f_{i_0}\dots f_{i_p}})$ of the component of α in $F(A_{f_{i_0}\dots f_{i_{k-1}}f_{i_{k+1}}\dots f_{i_p}})$ under the canonical map between these two groups.

Generalities on the Čech complex associated to a covering can be found in ([10], Chap. II, §5.1). Here we use the oriented version and have augmented the complex by $F(A)$.

We shall denote this Čech complex by $\overline{F(A)}_{f_0,\dots,f_r}$ or simply by $\overline{F(A)}$ if the elements $f_0,\dots,f_r$ are clear.

Let $f_0,\dots,f_r$ be elements of A. $0\leqq i_0<i_1\dots<i_p\leqq r$ be a set of natural numbers. By $A_{f_{i_0}\dots\hat{f}_{i_s}\dots f_{i_p}}$ we mean $A_{f_{i_0}\dots f_{i_{s-1}}f_{i_{s+1}}\dots f_{i_p}}$.

1.2. Theorem. *Let $F:\mathbf{CRg}\to\mathbf{Ab}$ be a functor, A a commutative ring, $f_0,\dots,f_r\in A$ a unimodular row. Assume that for all $0\leqq i_0<i_1<\dots<i_p\leqq r$ and $s\leqq p$ we have*

$$NF(A_{f_{i_0}\dots f_{i_s}\dots f_{i_p}}[X])\cong NF(A_{f_{i_0}\dots \hat{f}_{i_s}\dots f_{i_p}}[X])_{[f_{i_s}]}$$

then the Čech complex $\overline{NF(A)}_{f_0,\dots,f_r}$:

$$0\to NF(A)\to \prod_{0\leqq i\leqq r} NF(A_{f_i})\to \prod_{0\leqq i_0<i_1\leqq r} NF(A_{f_{i_0}f_{i_1}})\to\dots\to NF(A_{f_0\dots f_r})\to 0$$

is exact.

Proof. If one of the f_i (say f_0) is a unit then the complex is exact ([10], Chap. II, §5.7) because we can find a contracting homotopy (i.e. a homotopy between the identity and zero map of the complex) as follows.

$$\begin{tikzcd}
0 \arrow[r] & NF(A) \arrow[r, "d_0"] \arrow[d, shift right, "I"'] \arrow[d, shift left, "0"] & \prod\limits_{0\leqq i\leqq r} NF(A_{f_i}) \arrow[r, "d_1"] \arrow[d, shift right, "I"'] \arrow[d, shift left, "0"] \arrow[dl, "s_0"] & \prod\limits_{0\leqq i_0<i_1\leqq r} NF(A_{f_{i_0}f_{i_1}}) \arrow[r, "d_2"] \arrow[d, shift right, "I"'] \arrow[d, shift left, "0"] \arrow[dl, "s_1"] & \dots \arrow[r, "d_r"] \arrow[dl, "s_2"] & NF(A_{f_0\dots f_r}) \arrow[r] \arrow[d, shift right, "I"'] \arrow[d, shift left, "0"] & 0 \\
0 \arrow[r] & NF(A) \arrow[r, "d_0"] & \prod\limits_{0\leqq i\leqq r} NF(A_{f_i}) \arrow[r, "d_1"] & \prod\limits_{0\leqq i_0<i_1\leqq r} NF(A_{f_{i_0}f_{i_1}}) \arrow[r, "d_2"] & \dots \arrow[r, "d_r"] & NF(A_{f_0\dots f_r}) \arrow[r] & 0
\end{tikzcd}$$

with

$$(s_p(\alpha))_{i_0,\dots,i_{p-1}}=\begin{cases}(\alpha)_{0,i_0,\dots,i_{p-1}} & \text{if } i_0\neq 0\\ 0 & \text{if } i_0=0,\end{cases}$$

and $s_0(\alpha)=(\alpha)_0$.

Now we go to the general case.

For every $a\in A$ we can define a map $g_a:A[X]\to A[X,Y]$ by $g_a(q(X))=q(X+aY)$. This defines then a map $\psi_a:\overline{F(A[X])}\to\overline{F(A[X,Y])}$ between complexes. In particular we can take the difference $\psi_a-\psi_0$ which then induces a map between the following complexes

$$\begin{tikzcd}
0 \arrow[r] & NF(A) \arrow[r] \arrow[d] & \prod\limits_{0\leqq i\leqq r} NF(A_{f_i}) \arrow[r] \arrow[d] & \prod\limits_{0\leqq i_0<i_1\leqq r} NF(A_{f_{i_0}f_{i_1}}) \arrow[r] \arrow[d] & \dots \arrow[r] & NF(A_{f_0\dots f_r}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & NF(A[X]) \arrow[r] & \prod\limits_{0\leqq i\leqq r} NF(A_{f_i}[X]) \arrow[r] & \prod\limits_{0\leqq i_0<i_1\leqq r} NF(A_{f_{i_0}f_{i_1}}[X]) \arrow[r] & \dots \arrow[r] & NF(A_{f_0\dots f_r}[X]) \arrow[r] & 0
\end{tikzcd}$$

where

$$NF(A_{f_{i_0}\dots f_{i_s}}[X])=\operatorname{Ker}\left(F(A_{f_{i_0}\dots f_{i_s}}[X,Y])\xrightarrow{Y\mapsto 0}F(A_{f_{i_0}\dots f_{i_s}}[X])\right).$$

Let $H^i\overline{NF(A)}$ and $H^i\overline{NF(A[X])}$ be the cohomology groups of these complexes, and $\overline{\psi_a-\psi_0}$ the induced maps between the cohomology groups. If $\alpha(X)\in H^i\,\overline{NF(A)}$ then we shall denote by $\alpha(X+aY)-\alpha(X)$ its image in $H^i\,\overline{NF(A[X])}$ under $\overline{\psi_a-\psi_0}$. ($\alpha(X)$ is not a polynomial or something like that. We only use it for notational convenience, because it enables us to write $\alpha(aX+bY)$, by which we mean the image of $\alpha(X)$ under a homomorphism which we get by applying a functor, for instance cohomology, to the ring homomorphism $g:A[X]\to A[X,Y]$ with $g(q(X))=q(aX+bY)$.)

We have to prove that $\alpha(X)=0$. Write $I=\{a\in A|\alpha(X+aY)-\alpha(X)=0\}$. One easily checks that this is an ideal of A. (This kind of ideal is also used in a proof of the Serre problem by Vašerstein ([9], p. 204)). If we can show that $I=A$ then we can put $a=1$ and $Y=-X$ which shows that $\alpha(X)=\alpha(X-X)=\alpha(0)=1$. So it is enough to show for every $0\leqq i\leqq r$ that $f_i\in\sqrt{I}$. We shall do this for f_0.

If we denote by $\alpha_{f_0}(X)$ the image of $\alpha(X)$ in $H^i\overline{NF(A_{f_0})}$ then we have proved above that $\alpha_{f_0}(X)=0$. So $\alpha_{f_0}(X+Y)-\alpha_{f_0}(X)=0$ in $H^i\overline{NF(A_{f_0}[X])}$. By assumption we know that

$$NF(A_{f_0f_{i_0}\cdots f_{i_p}}[X])\cong NF(A_{f_{i_0}\cdots f_{i_p}}[X])_{[f_0]}.$$

But since localization commutes with taking homology we also have

$$(H^i\overline{NF(A[X])})_{[f_0]}=H^i\overline{NF(A[X])_{[f_0]}}$$

where $\overline{NF(A[X])_{[f_0]}}$ is the localized complex. So it follows that the image of $\alpha(X+Y)-\alpha(X)$ in $(H^i\overline{NF(A[X])})_{[f_0]}$ is zero. But since such a localization can be viewed as a direct limit (beginning of this section) it follows that there exists an s such that $\alpha(X+f_0^sY)-\alpha(X)=0$ in $H^iNF(A[X])$. So $f_0\in\sqrt{I}$, and we are finished.

1.3. The conditions

$$NF(A_{f_{i_0}\cdots f_{i_p}}[X])\cong NF(A_{f_{i_0}\cdots \hat{f}_{i_s}\cdots f_{i_p}}[X])_{[f_{i_s}]}$$

seem rather strange. But it will be shown in the next part that if $F=K_n$ for some n, then this condition always holds if for example A is reduced. We have stated this theorem in such a general form because we will derive some nice results from it for the NK_n-functors. Probably one can prove similar results for other functors by showing that the above conditions hold for these functors. To prove these conditions for K_n the main things we need are that K_n commutes with direct limits and an excision property ((*) in the proof of the lemma).

1.4. Lemma. *Let $f\in A$. If there exists a $g\in A$ such that $fg=0$ and $f+g$ is a non-zero divisor, then we have for all n that*

$$NK_n(A_f)\cong NK_n(A)_{[f]}.$$

Proof. Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n(A[X],XA[X]) & \longrightarrow & K_n(A[X]) & \longrightarrow & K_n(A) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n(A[X],XA[X]) & \longrightarrow & K_n(A[X]) & \longrightarrow & K_n(A) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where

$$K_n(A[X],XA[X])=\operatorname{Ker}\left(K_n(A[X])\xrightarrow{X\mapsto 0}K_n(A)\right)=NK_n(A)$$

and the vertical maps are defined by $X\mapsto fX$ for the lefthand side and middle maps and the identity for the righthand side maps. As described in the beginning of this section we can view $NK_n(A)_{[f]}$ as the direct limit of the lefthand side. But if we take the direct limit over the maps $A[X] \xrightarrow{X \mapsto fX} A[X]$ we get the ring $A+XA_f[X]$ and since K_n commutes with direct limits ([19], p. 20) we see that the direct limit of the middle maps is $K_n(A+XA_f[X])$. The direct limit of the righthand side is of course $K_n(A)$. So we get $NK_n(A)_{[f]} = \mathrm{Ker}(K_n(A+XA_f[X]) \to K_n(A))$. This kernel will be denoted by $K_n(A+XA_f[X], XA_f[X])$, a relative K_n-group. So it is enough to prove the following excision property

$$K_n(A+XA_f[X], XA_f[X]) \underset{\phi}{\xrightarrow{\sim}} K_n(A_f[X], XA_f[X]). \tag{$*$}$$

To prove this we state the following proposition which will also be used in Sect. 3 and is an immediate corollary of the work of Karoubi and Grayson.

1.5. Proposition. *Let $j: B \to B'$ be a ring homomorphism, $S \subset B$ a multiplicatively closed set of non-zero divisors, such that $j(S)$ are also non-zero divisors. If $B/sB \cong B'/j(s)B'$ for all $s \in S$ then we have two long exact sequences*

$$\begin{tikzcd}
\to K_n(\mathbf{H}(B)_S) \arrow[r] \arrow[d, "\cong"] & K_n(B) \arrow[r] \arrow[d] & K_n(B_S) \arrow[r] \arrow[d] & K_{n-1}(\mathbf{H}(B)_S) \to \arrow[d, "\cong"] \\
\to K_n(\mathbf{H}(B')_{j(S)}) \arrow[r] & K_n(B') \arrow[r] & K_n(B'_{j(S)}) \arrow[r] & K_{n-1}(\mathbf{H}(B')_{j(S)}) \to
\end{tikzcd}$$

where for all n the maps $K_n(\mathbf{H}(B)_S) \to K_n(\mathbf{H}(B')_{j(S)})$ are isomorphisms. ($\mathbf{H}(B)_S$ is the category of B-modules with homological dimension ≤ 1 and S-torsion.)

Proof. The proposition is an immediate consequence of the long exact localization sequences (see Grayson [11], p. 233) for B with respect to S and B' with respect to $j(S)$ and a lemma of Karoubi ([16], Appendix 5) which states that under the conditions of the proposition the categories $\mathbf{H}(B)_S$ and $\mathbf{H}(B')_{j(S)}$ are equivalent.

Now we can continue with the proof of 1.4 and shall prove the excision property ($*$) in two cases. First consider the case that f is a non-zero divisor. Then we can apply Proposition 1.5 with $B = A+XA_f[X]$, $B' = A$ and $S = (f^n)_{n \geq 0}$. Because now the maps $B \to B'$ and $B_S \to B'_{j(S)}$ split, we have that the kernels of the two other vertical maps are isomorphic which means exactly that ($*$) holds.

The second case is that in which f is idempotent. Then we have $A = A_f \times A_{(1-f)}$ is a direct product and since K_n commutes with direct products the excision property ($*$) is also evident in this case.

These two cases combine to yield the general case as follows. Using the conditions on f we see that $A_f = (A_{(f+g)})_{\frac{f}{f+g}}$ (i.e. two localizations after each other). $\frac{f}{f+g}$ an idempotent element of $A_{(f+g)}$ and $f+g$ is a non-zero divisor. Hence we have

$$\begin{aligned}
NK_n(A_f) &\cong NK_n\left((A_{(f+g)})_{\frac{f}{f+g}}\right) \cong NK_n(A_{(f+g)})_{\left[\frac{f}{f+g}\right]} \\
&\cong (NK_n(A)_{[f+g]})_{\left[\frac{f}{f+g}\right]} \cong NK_n(A)_{[f]}.
\end{aligned}$$

The last isomorphism can be proved by viewing $NK_n(A)$ as a $\mathbb{Z}[T,U]/(TU-U^2)$-module where T acts through the map $X \mapsto (f+g)X$ and U acts through the map

$X \mapsto fX$. Then it follows from commutative algebra that

$$(NK_n(A)_{[f+g]})\left[\frac{f}{f+g}\right] = (NK_n(A)_T)_{\frac{U}{T}} = NK_n(A)_{TU} = NK_n(A)_{U^2}$$
$$= NK_n(A)_{[f]},$$

where for example $NK_n(A)_{TU}$ means localization of $NK_n(A)$ with respect to the multiplicatively closed set $(TU)^n_{n\in\mathbb{N}}$. So we have finished the proof of Lemma 1.4.

The following lemma shows that the condition in Lemma 1.4 is not a very strong condition.

1.6. Lemma. *Let A be a reduced noetherian ring. Let $f \in A$. Then there exist a $g \in A$ such that $gf = 0$ and $g+f$ is a non-zero divisor.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A which contain the element f. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the other minimal prime ideals. (Because A is noetherian there are only finitely many minimal prime ideals.) $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r \cap \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s = 0$ and for every $1 \leqq i \leqq r$ we can find a $g_i \notin \mathfrak{p}_i$ but contained in all other minimal prime ideals. Therefore $g = \sum_{i=1}^{r} g_i \in \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ and $g \notin \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$. Hence

$$g \cdot f \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r \cap \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$$

and $g+f$ is not contained in any minimal prime ideal, so $f+g$ is a non-zero divisor.

1.7. Corollary. *Let A be a reduced ring, $n \geqq 0$.*

(i) *If $f \in A$ then we have $NK_n(A_f) \cong NK_n(A)_{[f]}$.*

(ii) *If $f_0, \ldots, f_r \in A$ is a unimodular row then we have that the complex*

$$0 \to NK_n(A) \to \prod_{0 \leqq i \leqq r} NK_n(A_{f_i}) \to \prod_{0 \leqq i_0 < i_1 \leqq r} NK_n(A_{f_{i_0} f_{i_1}}) \to \ldots \to NK_n(A_{f_0 \ldots f_r}) \to 0$$

is exact.

Proof. (i) In the proof of Lemma 1.4 we have shown that $NK_n(A_f) \cong NK_n(A)_{[f]}$ is equivalent to the excision property

$$K_n(A + XA_f[X], XA_f[X]) \xrightarrow{\simeq} K_n(A_f[X], XA_f[X]). \qquad (*)$$

A is the direct limit of its finitely generated subrings which contain the element f. Since K_n commutes with direct limits it is enough to prove that $(*)$ holds for finitely generated rings over $\mathbb{Z}$. So we may assume that A is noetherian. But then we can use 1.6 and 1.4 and we are done.

(ii) Follows directly from (i) and Theorem 1.2.

1.8. Remark. We can give another proof of Corollary 1.7(ii) which doesn't use Theorem 1.2, but only Corollary 1.7(i).

Let Big $W(A) = (1 + TA[\![T]\!])^*$ be the big Witt vectors on A. (For the notations and calculations we use, see [4], I, § 1). In ([5], § 2) Bloch states that $NK_n(A)$ can be given a Big $W(A)$-module structure. The multiplication by $\omega(1-fT)$ on $NK_n(A)$ is induced by the map which sends T to fT. (See Stienstra, [20].) So this shows that $NK_n(A)_{[f]}$ in our notation is in fact the module $NK_n(A)_{\omega(1-fT)}$ (i.e. localization of $NK_n(A)$ with respect to powers of $\omega(1-fT)$.) Hence if $f \in A$ fulfils the condition of

1.4 we have

$$NK_n(A_f) \cong NK_n(A)_{\omega(1-fT)}. \qquad (**)$$

The alternative proof of Corollary 1.7 now goes as follows. Let $\widetilde{\mathcal{N}\mathcal{K}_n(A)}$ be the quasi-coherent sheaf on Spec(Big $W(A)$) associated to the Big $W(A)$-module $NK_n(A)$ (see [13], Chap. I, § 1.3). If we can show that $\omega(1-f_0T)$, $\omega(1-f_1T), \ldots, \omega(1-f_rT)$ form a unimodular row in Big $W(A)$, we know that the $D(\omega(1-f_iT))_{0 \leqq i \leqq r}$ give an open covering of Spec(Big $W(A)$). Then it is known that the cohomology of the Čech-complex associated to this covering with coefficients in $\widetilde{\mathcal{N}\mathcal{K}_n(A)}$ is trivial ([14], Chap. III, 1.2.4). But this Čech-complex is exactly the complex of the corollary by using (**).

So we only have to prove that $\omega((1-f_0T)^{-1}), \ldots, \omega((1-f_rT)^{-1})$ form a unimodular row, because these are the negatives of the $\omega(1-f_iT)$. Hence it is enough to prove that we can find g_{ij} with $0 \leqq i \leqq r$ and $1 \leqq j < \infty$ such that for all n we have

$$\sum_{i=0}^{r} \left(\sum_{j=1}^{n} \omega((1-f_iT)^{-1}) \cdot \omega((1-g_{ij}T^j)^{-1}) \right) = \omega(1-T+T^{n+1}(a_{n+1}+\ldots)). \qquad (***)$$

So assume we have found g_{ij} for $1 \leqq j \leqq n$ such that (***) holds. Then for $g_{i,n+1} \in A$ with $0 \leqq i \leqq r$ we must have

$$\begin{aligned}
&\sum_{i=0}^{r} \left(\sum_{j=1}^{n+1} \omega((1-f_iT)^{-1}) \cdot \omega((1-g_{ij}T^j)^{-1}) \right) \\
&= \omega(1-T+T^{n+1}a_{n+1}+T^{n+2}(a_{n+2}+\ldots)) + \sum_{i=0}^{r} \omega((1-f_i^{n+1}g_{i(n+1)}T^{n+1})^{-1}) \\
&= \omega(1-T+T^{n+1}a_{n+1}+T^{n+2}(a_{n+2}+\ldots)) \\
&\quad + \sum_{i=0}^{r} \omega(1+f_i^{n+1}g_{i(n+1)}T^{n+1}+T^{n+2}(\ldots)) \\
&= \omega\left(1-T+T^{n+1}\left(a_{n+1}+\sum_{i=0}^{r} f_i^{n+1}g_{i(n+1)}\right)+T^{n+2}(\ldots)\right) \\
&= \omega(1-T+T^{n+2}(\ldots)).
\end{aligned}$$

Then we have to solve the equation $a_{n+1} + \sum_{i=0}^{r} f_i^{n+1} g_{i(n+1)} = 0$. But this is always possible, because $f_0^{n+1}, \ldots, f_r^{n+1}$ form a unimodular sequence in A.

1.9. Corollary. *Let A be a commutative ring. Then for all $n \geqq 0$ we have*

(i) *If $S \subset A$ is a multiplicatively closed set of non-zero divisors then $NK_n(A)=0$ implies $NK_n(A_S)=0$.*

If moreover A is reduced we have

(ii) *$NK_n(A)=0$ implies $NK_n(A_\mathfrak{p})=0$ for every prime ideal $\mathfrak{p} \subset A$.*

(iii) *The map $NK_n(A) \to \prod_{\substack{\mathfrak{m} \subset A \\ \mathfrak{m}\ \text{maximal ideal}}} NK_n(A_\mathfrak{m})$ is injective.*

Proof. (i) Take $f \in S$ then $NK_n(A)=0$ implies $NK_n(A)_{[f]}=0$. So the statement follows from 1.4 and the fact that K-theory commutes with direct limits.

(ii) as (i).

(iii) Take $\alpha \in NK_n(A)$ such that for every maximal ideal $\mathfrak{m} \subset A$ we have that $\alpha_{\mathfrak{m}}$, the image of α in $NK_n(A_{\mathfrak{m}})$, is zero. Again using the direct limit argument it follows that for every maximal ideal $\mathfrak{m}$ one can find an $f \notin \mathfrak{m}$ such that α_f is zero. But one can take finitely many of these f in such a way that they form a unimodular row. Now use the injectivity of the map $NK_n(A) \to \prod NK_n(A_f)$ from Corollary 1.7(ii) to show that α is zero.

1.10. The remaining part of this section will not be used in the rest of this paper.

In the following we assume that A is reduced. Let NK_n be the presheaf associated with NK_n as described in the beginning of this section. Let $\mathscr{N}\mathscr{K}_n$ be the sheafification of NK_n. Then using Corollary 1.7(ii) it follows that

$$\mathscr{N}\mathscr{K}_n(D(f)) = \mathsf{NK}_n(D(f)) = NK_n(A_f).$$

Now let $H^q(A, \mathscr{N}\mathscr{K}_n)$ be the cohomology of $\operatorname{Spec}(A)$ with coefficients in $\mathscr{N}\mathscr{K}_n$ and $\check{H}^q(A, \mathscr{N}\mathscr{K}_n)$ the corresponding Čech-cohomology (see [10], Chap. II, § 5). Quillen ([19], p. 53) has shown that for a regular scheme X of finite type over a field $H^p(X, \mathscr{K}_p) \cong A^p(X)$ (i.e. the cycles of codimension p modulo rational equivalence). We shall show that for a reduced ring the $\mathscr{N}\mathscr{K}$-sheaves give trivial cohomology.

1.11. Corollary. *If A is a reduced ring, then for all $n \geqq 0$ and $q \geqq 1$ we have $H^q(A, \mathscr{N}\mathscr{K}_n) = \check{H}^q(A, \mathscr{N}\mathscr{K}_n) = 0$.*

Proof. If $f_0, \ldots, f_r \in A$ form a unimodular row, then

$$\operatorname{Spec}(A) = D(f_0) \cup D(f_1) \cup \ldots \cup D(f_r).$$

So we have an open covering of $\operatorname{Spec}(A)$. But by Corollary 1.7(ii) for this covering the Čech-cohomology with coefficients in $\mathscr{N}\mathscr{K}_n$ is trivial. Further, coverings of this kind form a cofinal subsystem of all open coverings. Hence $\check{H}^q(A, \mathscr{N}\mathscr{K}_n) = 0$ for all $q \geqq 1$.

Let $\mathfrak{U}$ be the family of all open subsets $D(f)$ of $\operatorname{Spec}(A)$. This is a covering of $\operatorname{Spec}(A)$ with $D(f) \cap D(g) = D(fg)$. For every open set $U \subset \operatorname{Spec}(A)$ and point $p \in U$ we can find an f with $p \in D(f) \subset U$. Further we have $\check{H}^q(D(f), \mathscr{N}\mathscr{K}_n) = \check{H}^q(A_f, \mathscr{N}\mathscr{K}_n) = 0$ for all f. Hence we can apply a theorem of H. Cartan ([10], Chap. II, 5.9.2), which says that $H^q(A, \mathscr{N}\mathscr{K}_n) \cong \check{H}^q(A, \mathscr{N}\mathscr{K}_n) = 0$.

1.12. Remark. For NK_n with $n = 0, 1, 2$ one can prove Lemma 1.4 without any restriction on f. This can be seen as follows. One has to prove the excision property

$$K_n(A + XA_f[X], XA_f[X]) \cong K_n(A_f[X], XA_f[X]).$$

For K_0 this is true because K_0-excision always holds ([2]). For K_1 one can use ([21], 2.5) which says that it is enough to show that $\Omega_{A_f|A} = 0$. But this follows from $D\left(\frac{a}{f^n}\right) = \frac{1}{f^n} Da - \frac{a}{f^{2n}} Df^n = 0$. For K_2 it will be proved by Van der Kallen in the appendix. In fact a slightly more general excision property can be proved for K_1 and K_2 (see appendix, the cases proved there for K_2 also hold for K_1).

From this it follows that for $n=0,1$ or 2 the Corollaries 1.7, 1.9 and 1.11 also hold in the non-reduced case. In particular in this form it shows that Corollary 1.9(ii), (iii) is an extension of ([21], 1.1).

1.13. Now we shall give similar results for other functors related with algebraic K-theory.

For every $p \geqq 0$ one can define the functors

$$C_pK_n: \mathbf{CRg} \to \mathbf{Ab} \quad \text{by}$$

$$C_pK_n(A) = \mathrm{Ker}\,(K_n(A[X]/X^{p+1}) \xrightarrow{X \mapsto 0} K_n(A)).$$

Bloch (see [4], II, § 1) has introduced these functors and called them curves on K_n.

Let $f \in A$. As at the beginning of Sect. 1, we can use the ring homomorphism $\phi_f: A[X]/X^{p+1} \to A[X]/X^{p+1}$ with $\phi_f(q(X)) = q(fX)$ to give $C_pK_n(A)$ a $\mathbb{Z}[T]$-module structure. And again we define $C_pK_n(A)_{[f]}$ to be the localization of $C_pK_n(A)$ with respect to $(T^n)_{n \geqq 0}$.

By going through the proof of Lemma 1.4 one easily sees that this lemma also holds if we replace NK_n by C_pK_n. As in Remark 1.8 we can also give $C_pK_n(A)$ a Big $W(A)$-module structure (see [4], II, § 2) and then Lemma 1.4 can be restated as follows.

If $f \in A$ is such that there exists a $g \in A$ with $fg=0$ and $f+g$ a non-zero divisor, then for all $n \geqq 0$ and $p \geqq 0$ we have

$$C_pK_n(A_f) \cong C_pK_n(A)_{\omega(1-fT)}.$$

Corollaries 1.7(ii) and 1.9 also hold for the functor C_pK_n. For 1.9 this is immediate from the C_pK_n-versions of 1.4 and 1.7(ii). A proof of 1.7(ii) can be given similar to the proof of the statement of Theorem 1.2. In that proof $g_a: A[X] \to A[X, Y]$ has to be replaced by $g'_a: A[X]/X^{p+1} \to A[X, Y]/(X, Y)^{p+1}$ with $g'_a(q(X)) = q(X + aY)$ and one has to use that

$$\begin{aligned} &\mathrm{Ker}\,(K_n(A_{f_0 f_{i_0} \dots f_{i_t}}[X, Y]/(X, Y)^{p+1}) \xrightarrow{Y \mapsto 0} K_n(A_{f_0 f_{i_0} \dots f_{i_t}}[X]/X^{p+1})) \\ &\quad \cong \mathrm{Ker}\,(K_n(A_{f_{i_0} \dots f_{i_t}}[X, Y]/(X, Y)^{p+1}) \xrightarrow{Y \mapsto 0} K_n(A_{f_{i_0} \dots f_{i_t}}[X]/X^{p+1}))_{[f_0]} \end{aligned}$$

where as before the last group is again a localization induced by the map $Y \to f_0 Y$. The rest of the proof is a straightforward inspection.

We can give an alternative proof of 1.7(ii) by using that $A[X]/X^{p+1}$ is a graded ring. More generally let $B = \bigoplus_{i \geqq 0} B_i$ be a graded ring. Let π be the projection on the zero-th component. Define

$$N^+K_n(B) = \mathrm{Ker}\,(K_n(B) \xrightarrow{K_n(\pi)} K_n(B_0)).$$

Let $f_0, \dots, f_r \in B_0$ be a unimodular row. Then again we can form a Čech-complex

$$\begin{aligned} 0 \to N^+K_n(B) \to &\prod_{0 \leqq i \leqq r} N^+K_n(B_{f_i}) \\ &\to \prod_{0 \leqq i_0 < i_1 \leqq r} N^+K_n(B_{f_{i_0} f_{i_1}}) \to \dots \to N^+K_n(B_{f_0 \dots f_r}) \to 0. \end{aligned}$$

Denote this complex by $\overline{N^+K_n(B)}$. We shall prove that $\overline{N^+K_n(B)}$ is a direct summand of the complex $\overline{NK_n(B)}$. For this we use the following ring homomorphism introduced by Weibel ([22], 1.8).

$$\varrho: B \to B[T] \quad \text{with} \quad \varrho(a_0 + a_1 + \dots + a_m) = a_0 + a_1 T + a_2 T^2 + \dots + a_m T^m$$

if $a_i \in B_i$ is the i-th component. This clearly induces a map $\overline{N^+K_n(\varrho)}: \overline{N^+K_n(B)} \to \overline{NK_n(B)}$ between complexes. Now define $\theta: B[T] \to B$ by $\theta(f(T)) = f(1)$. Then we get the map $\overline{K_n(\theta)} - \overline{K_n(i \circ \pi \circ \theta)}: \overline{K_n(B[T])} \to \overline{K_n(B)}$ between complexes, where $i: B_0 \to B$ is the natural injection. One easily sees that this induces a map $\chi: \overline{NK_n(B)} \to \overline{N^+K_n(B)}$ such that $\chi \circ \overline{N^+K_n(\varrho)} = Id$. So $\overline{N^+K_n(B)}$ is a direct summand of $\overline{NK_n(B)}$. Now we can apply this to $B = A[X]/X^{p+1}$. This is a graded ring and $\overline{N^+K_n(B)} = \overline{C_pK_n(A)}$. For 1.7(ii) we may assume that A is reduced. Let $f_0, \dots, f_r \in A$ be unimodular. The complex $\overline{C_pK_n(A)}_{f_0, \dots f_r}$ is a direct summand of $\overline{NK_n(B)}_{f_0, \dots, f_r}$. So it is enough to show that the latter complex is exact. As in the proof of 1.7(i) we may assume that A is noetherian. By Lemma 1.4 it is enough to show that we can find for every $f \in A$ a $g \in B$ such that $fg = 0$ and $f + g$ is a non-zero divisor. By Lemma 1.5 such a g can be found in A and $f + g$ a non-zero divisor in A implies $f + g$ a non-zero divisor in B. Hence we are finished.

1.14. Next we consider the functors EK_n (see [12]) defined by Grayson. Let A be a commutative ring. A^* the units of A, Define $A\{X\} = S^{-1}A[X]$ where $S = \{A^* + XA[X]\}$. Sending X to zero gives a split ringhomomorphism $A\{X\} \to A$. This induces a split homomorphism $K_n(A\{X\}) \to K_n(A)$. Now define $EK_n(A)$ to be the kernel of this map. Grayson ([12]) has shown that these functors are related to the K-theory of the category of endomorphisms over the ring A. Also for these functors 1.4, 1.7 and 1.9 hold. The proofs are similar to those for NK_n.

2. On some Questions of Bass

A ring A is called K_n-regular if for all r we have $K_n(A) \cong K_n(A[T_1, \dots, T_r])$. Bass ([3], problem III) has raised the following three related questions.

(a) Does K_1-regularity imply K_0-regularity?

(b) Does $NK_1(A) = 0$ imply $NK_0(A) = 0$?

(c) Define

$$f: K_0(A[T]) \to K_1(A[T, T^{-1}])$$

by $f = (\cdot T) \circ K_0(\delta)$ where $\delta: A[T] \to A[T, T^{-1}]$ is the natural inclusion and

$$(\cdot T): K_0(A[T, T^{-1}]) \to K_1(A[T, T^{-1}])$$

is the multiplication by $T \in K_1(A[T, T^{-1}])$. The question is whether f is injective.

It is clear that a positive answer to (b) implies a positive answer to (a). Bass ([3], problem III) has also shown that a positive answer to (c) implies a positive answer to (b). So these questions are strongly related.

Now we can state that question (a) has a positive answer as can be seen from the following more general corollary of 1.9.

2.1. Corollary *(Van der Kallen). For all $n \geq 1$ we have*
(i) *$NK_n(A[T])=0$ implies $NK_{n-1}(A)=0$.*
(ii) *K_n-regularity implies K_{n-1}-regularity.*

Proof. (ii) immediately follows from (i). So it is enough to prove (i). Consider the following exact localization sequence

$$0 \to NK_n(A) \to NK_n(A[T]) \oplus NK_n(A[T^{-1}]) \to NK_n(A[T, T^{-1}]) \to NK_{n-1}(A) \to 0.$$

One finds this NK_n-sequence as the kernel of the corresponding K_n-sequences for $A[X]$ and A ([11], p. 237). But using Corollary 1.9(i) we see that $NK_n(A[T])=0$ implies $NK_n(A[T, T^{-1}])=0$ and the exact sequence shows that $NK_{n-1}(A)=0$. q.e.d.

Concerning question (c) we shall now give an example which shows that f is not always injective.

2.2. Counterexample. Let $\mathbb{F}_2$ be the field of two elements. Let A be the pullback in the following diagram.

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{F}_2[X] \arrow[d, "j_1"] \\ \mathbb{F}_2[X] \arrow[r, "j_2"] & \mathbb{F}_2[X]/X^3 \end{tikzcd}$$

i.e. $A=\{(a,b) \in \mathbb{F}_2[X] \times \mathbb{F}_2[X] \mid a \equiv b \bmod (X^3)\}$, j_1, j_2 are the canonical projection maps, i_1 is the projection onto the first coordinate and i_2 is the projection onto the second coordinate.

In ([18], §3 and §6) it is proved that there exists a Mayer-Vietoris sequence for this diagram and for the diagram which we get by adding to each ring a polynomial variable. So we get a Mayer-Vietoris sequence

$$\begin{aligned} &NK_2(\mathbb{F}_2[X]) \oplus NK_2(\mathbb{F}_2[X]) \to NK_2(\mathbb{F}_2[X]/X^3) \xrightarrow{\partial_1} NK_1(A) \\ &\to NK_1(\mathbb{F}_2[X]) \oplus NK_1(\mathbb{F}_2[X]) \to NK_1(\mathbb{F}_2[X]/X^3) \xrightarrow{\partial_0} NK_0(A) \\ &\to NK_0(\mathbb{F}_2[X]) \oplus NK_0(\mathbb{F}_2[X]) \to NK_0(\mathbb{F}_2[X]/X^3). \end{aligned}$$

We shall use notations and constructions of ([18]) without any further reference. Let $x \in \mathbb{F}_2[X]/X^3$ be the residue class of X. Take $(1+x^2T) \in NK_1(\mathbb{F}_2[X]/X^3)$. Clearly $1+x^2T \neq 0$. Now we can calculate $\partial_0(1+x^2T)=[P]-[A[T]]$ where

$$P=M(\mathbb{F}_2[X,T], \mathbb{F}_2[X,T], 1+x^2T).$$

Take

$$Q=M(\mathbb{F}_2[X,T], \mathbb{F}_2[X,T], 1-x^2T).$$

Take

$$\bar{\alpha}=\begin{pmatrix} 1+x^2T & 0 \\ 0 & 1+x^2T \end{pmatrix} \in GL_2(\mathbb{F}_2[X,T]/X^3)$$

and

$$\alpha=\begin{pmatrix}1+X^2T & X^3T^2\\ X^3T & 1+X^2T+X^4T^2\end{pmatrix}\in GL_2(\mathbb{F}_2[X,T])$$

a lifting of $\bar{\alpha}$. Then we can form the following commutative diagram:

$$\begin{tikzcd}
P\oplus Q \arrow[rrr] \arrow[dr, "\cong", "\phi"'] \arrow[ddd] & & & (\mathbb{F}_2[X,T])^2 \arrow[dl, "\alpha"'] \arrow[dd] \\
& (A[T])^2 \arrow[r] \arrow[ddl] & (\mathbb{F}_2[X,T])^2 \arrow[dd] & \\
& & & (\mathbb{F}_2[X,T]/X^3)^2 \arrow[dl, "\bar{\alpha}"'] \\
(\mathbb{F}_2[X,T])^2 \arrow[rr] & & (\mathbb{F}_2[X,T]/X^3)^2 &
\end{tikzcd} \tag{$*$}$$

where ϕ is an isomorphism of $A[T]$-modules.

Since $\mathbb{F}_2[X]$ is regular we have that $NK_n(\mathbb{F}_2[X])=0$. Hence ∂_0 and ∂_1 are isomorphisms. Therefore $\partial_0(1+x^2T)\neq 0$ and so $[P]\neq[A[T]]$.

Our counterexample will be the ring A and the element $([P]-[A[T]])\in K_0(A[T])$. We can tensor diagram ($*$) with $\mathbb{F}_2[T,T^{-1}]$ over $\mathbb{F}_2[T]$. We shall denote the new modules and homomorphisms by a tilde. Then we have

$$f([P])=\tilde{\phi}\begin{pmatrix}TI_{\tilde{P}} & 0\\ 0 & I_{\tilde{Q}}\end{pmatrix}\tilde{\phi}^{-1}.$$

Because $A[T,T^{-1}]\subset\mathbb{F}_2[X][T,T^{-1}]\times\mathbb{F}_2[X][T,T^{-1}]$ we can split each entry of the matrix $f([P])$ and then split the matrix into two components, which gives

$$f([P])=\left(\begin{pmatrix}T & 0\\ 0 & 1\end{pmatrix},\tilde{\alpha}\begin{pmatrix}T & 0\\ 0 & 1\end{pmatrix}\tilde{\alpha}^{-1}\right).$$

Similarly we can calculate $f([A[T,T^{-1}]])$ and we get

$$f([P])f([A[T]])^{-1}=\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\tilde{\alpha}\begin{pmatrix}T & 0\\ 0 & 1\end{pmatrix}\tilde{\alpha}^{-1}\begin{pmatrix}T & 0\\ 0 & 1\end{pmatrix}^{-1}\right).$$

Consider the Steinberg symbol $\{1+x^2T,T\}\in K_2(\mathbb{F}_2[X,T,T^{-1}]/X^3)$. We claim that $\partial_1^*\{1+x^2T,T\}=f([P])f([A[T]])^{-1}$, where

$$\partial_1^*: K_2(\mathbb{F}_2[X,T,T^{-1}]/X^3)\to K_1(A[T,T^{-1}])$$

is the boundary map of the Mayer-Vietoris sequence. First consider the boundary map

$$\partial: K_2(\mathbb{F}_2[X,T,T^{-1}]/X^3)\to K_1(\mathbb{F}_2[X,T,T^{-1}],X^3)$$

(i.e. the relative K_1). A description of this map is given on p. 54 in Milnor's book.

If one uses the canonical splitting of the ring homomorphism

$$\mathbb{F}_2[X,T,T^{-1}]\to\mathbb{F}_2[X,T,T^{-1}]/X^3$$

one easily shows that $\partial\{1+x^2T, T\}$ is given by the matrix

$$\left[\begin{pmatrix} 1+X^2T+X^4T^2+X^6T^3 & X^4T^2 \\ X^4T^2 & 1+X^2T \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}\right] \in \mathrm{GL}_2(\mathbb{F}_2[X,T,T^{-1}],X^3).$$

But since

$$\begin{pmatrix} 1+X^2T+X^4T^2+X^6T^3 & X^4T^2 \\ X^4T^2 & 1+X^2T \end{pmatrix} \cdot \begin{pmatrix} 1+X^2T & X^3T^2 \\ X^3T & 1+X^2T+X^4T^2 \end{pmatrix}^{-1} \in \mathrm{GL}_2(\mathbb{F}_2[X,T,T^{-1}],X^3),$$

it follows that

$$\partial\{1+x^2T, T\} = \left[\begin{pmatrix} 1+X^2T & X^3T^2 \\ X^3T & 1+X^2T+X^4T^2 \end{pmatrix}, \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}\right].$$

From this it easily follows ([18], p. 55) that

$$\partial_1^*\{1+x^2T, T^{-1}\} = f([P])f([A[T]])^{-1}.$$

So to finish our example it is enough to show that $\{1+x^2T, T\}=0$. In the Dennis-Stein notation ([7], §1) we have $\{1+x^2T, T\} = \langle x^2, T\rangle$. But

$$\langle x^2, T\rangle^{-1} = \langle T, x^2\rangle = \langle xT, x\rangle^2 = \langle xT, x+x+x^3T\rangle = \langle xT, x^3T\rangle = 0$$

(see [7], [17]). q.e.d.

Unfortunately we haven't found an answer to question (b).

3. NK_2 of 1-Dimensional Rings

3.1. Let B be a ring, $S \subset B$ a multiplicatively closed set of non-zero divisors. Consider the following diagram of rings and homomorphisms.

$$\begin{array}{ccc} B & \longrightarrow & B_S \\ {\scriptstyle j}\downarrow & & \downarrow \\ \hat{B} & \longrightarrow & \hat{B}_S \end{array}$$

where $\hat{B} = \varprojlim_{s\in S} B/sB$, $B_S = S^{-1}B$, $j: B \to \hat{B}$ the natural ring homomorphism and $\hat{B}_S = j(S)^{-1}\hat{B}$.

Karoubi ([16]) introduced this kind of diagrams and one easily sees that for this homomorphism $j: B \to \hat{B}$ and $S \subset B$ the conditions of Proposition 1.5 hold.

Hence we have the following diagram with long exact sequences

$$\begin{array}{ccccccccc} \ldots \to & K_n(\mathbf{H}(B)_S) & \to & K_n(B) & \to & K_n(B_S) & \to & K_{n-1}(\mathbf{H}(B)_S) & \to \ldots \\ & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & \\ \to & K_n(\mathbf{H}(\hat{B})_S) & \to & K_n(\hat{B}) & \to & K_n(\hat{B}_S) & \to & K_{n-1}(\mathbf{H}(\hat{B})_S) & \to \ldots \end{array} \qquad (*)$$

From this diagram one easily gets a long exact Mayer-Vietoris sequence

$$\dots K_n(B) \to K_n(\hat{B}) \oplus K_n(B_S) \to K_n(\hat{B}_S) \to K_{n-1}(B) \to \dots$$

This sequence gives a solution to a problem of Bak ([1]).

In this section we shall use the diagram (∗) in the following three cases.

Case 1. Let $B=A$ be a 1-dimensional noetherian local ring with maximal ideal $\mathfrak{m}$. Take $s \in \mathfrak{m}$ a non-zero divisor. Let $S=\{s^n\}_{n\in\mathbb{N}}$. Since $\mathfrak{m}$ is the only prime ideal which contains s, there exists an r such that $\mathfrak{m}^r \subset sA$. Hence $\hat{A} = \varprojlim A/s^l A = \varprojlim A/\mathfrak{m}^l A$. So $\hat{A}$ is the $\mathfrak{m}$-adic completion of A.

Case 2. Let A, S and $\mathfrak{m}$ be as in Case 1. Take $B=A[T_1,\dots,T_p]$ a polynomial extension of A. So $\hat{B}=\hat{A}[T_1,\dots,T_p]$, $\hat{B}_S=\hat{A}_S[T_1,\dots,T_p]$. Hence we also have a diagram (∗) in this case.

If one furthermore supposes that the ring A and its $\mathfrak{m}$-adic completion $\hat{A}$ are reduced, one sees that A_S and $\hat{A}_S$ are 0-dimensional reduced rings. Hence A_S and $\hat{A}_S$ are products of fields. Therefore we can combine Case 1 and Case 2 as follows.

Let $N_pK_n(A)=\operatorname{Ker}(K_n(A[T_1,\dots,T_p]) \to K_n(A))$. The diagrams of Case 1 and Case 2 give rise to a diagram for N_pK_n, and since $N_pK_n(A_S)=N_pK_n(\hat{A}_S)=0$ one easily sees that $N_pK_n(A)=N_pK_n(\hat{A})$ for all $p \geqq 1$ and $n \geqq 0$.

Case 3. Let k be a field and $K=\prod_{i=1}^{s} k_i$ be a product of finite field extensions of k. Let $B=\{f\in K[X] \mid f(0)\in k\}$. Take $S=(X^n)_{n\in\mathbb{N}}$. Then $\hat{B}=\{f\in K[\![X]\!] \mid f(0)\in k\}$. As in Case 2 we can adjoin polynomial variables to B and then get a diagram for N_pK_n. Since both B_S and $\hat{B}_S$ are regular we again have as in Case 2 that $N_pK_n(B) = N_pK_n(\hat{B})$.

3.2. Let A be a 1-dimensional noetherian local ring. As remarked after Case 2 we have that $N_pK_n(A)=N_pK_n(\hat{A})$ if both A and $\hat{A}$ are reduced (A is then called analytically reduced). If A is universellement japonnais we even know that A is reduced implies that $\hat{A}$ is reduced ([15], 7.6.4 and 7.7.2).

In the next proposition we shall show that if A is a reduced seminormal 1-dimensional local ring such that $\bar{A}$ (i.e. the normalization of A) is a finite A-module, then A is analytically reduced. For the definition of seminormal 1-dimensional local rings see for example [21].

3.3. Proposition. *Let A be a 1-dimensional noetherian reduced seminormal local ring such that $\bar{A}$ is a finite A-module. Then for all $n \geqq 0$ and $p \geqq 1$ we have $N_pK_n(A) \cong N_pK_n(\hat{A})$.*

Proof. It is enough to show that $\hat{A}$ is reduced. Let $\mathfrak{m}_1,\dots,\mathfrak{m}_s$ be the maximal ideals of $\bar{A}$ and $J=\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$ the Jacobson radical of $\bar{A}$. So $J=\mathfrak{m}$. Hence we have injections $A/\mathfrak{m}^n \to \bar{A}/J^n$, which implies that $\varprojlim A/\mathfrak{m}^n \to \varprojlim \bar{A}/J^n$ is injective. But $\bar{A}$ is regular semilocal and so $\varprojlim \bar{A}/J^n$ is a product of regular complete local rings. From this we see that $\varprojlim A/\mathfrak{m}^n = \hat{A}$ has no nilpotents.

3.4. Theorem. *Let A be a 1-dimensional reduced noetherian local ring, such that $\bar{A}$ is a finite A-module. Assume that A is seminormal and equicharacteristic. Then A is K_2-regular implies A is regular.*

Proof. A is regular if and only if $\hat{A}$ is regular. A is seminormal if and only if $\hat{A}$ is seminormal (see e.g. [6]). Since also the finiteness condition on $\bar{A}$ still holds after completion we may assume by Proposition 3.3 that A is complete. But then Davis ([6], §3) has proved that $A=\{f\in K[\![X]\!]|f(0)\in k\}$ where $K=\bar{A}/J=\prod k_i$ with $k_i=\bar{A}/\mathfrak{m}_i$ are finite field extensions of $k=A/\mathfrak{m}$. We know by Case 3 of 3.1 that $N_pK_2(A)=N_pK_2(B)$ if $B=\{f\in K[X]|f(0)\in k\}$. Hence it is easy to see that we may assume that $A\cong\{f\in K[X]|f(0)\in k\}$. Therefore it is enough to show that A is not K_2-regular in the following three cases.

I. $\bar{A}$ has more than one maximal ideal.

II. $\bar{A}$ has one maximal ideal $\mathfrak{m}_1$, but $k\subsetneqq k_1$ is an inseparable field extension.

III. $\bar{A}$ has one maximal ideal $\mathfrak{m}_1$, but $k\subsetneqq k_1$ is a separable field extension.

Case I. $\bar{A}=\prod_{i=1}^{s} k_i[X_i]$ and

$$A=\{(f_1,\dots,f_s)\in\bar{A}|f_1(0)=\dots=f_s(0)\in k\}.$$

Let $L\supset k$ be a finite extension field of k such that $k_i\subset L$ for all i. Let

$$B=\{(g_1,\dots,g_s)\in\prod_{i=1}^{s} L[X_i]|g_1(0)=\dots=g_s(0)\}.$$

Hence $B\cong L[X_1,\dots,X_s]/(X_iX_j|i\neq j)$. Now we can use a result of Dennis and Krusemeyer ([8], §4) which says that if R is a regular ring and

$$B_R=R[X_1,\dots,X_s]/(X_iX_j|i\neq j)$$

then

$$K_2(B_R)\cong K_2(R)\oplus(R^+)^n$$

where R^+ is the additive group of R and $n=\binom{s}{2}$ the binomial coefficient. This isomorphism can be arranged in such a way that $(a,0,\dots,0)\in(R^+)^n$ corresponds with the Dennis-Stein symbol $\langle aX_1,X_2\rangle$.

We apply this to $R=L[T]$. Clearly we have an injection $A\to B$. This induces a map $NK_2(A)\to NK_2(B)$. Take the symbol $\langle TX_1,X_2\rangle\in NK_2(A)$. Its image in $NK_2(B)\subset K_2(B[T])$ corresponds with $(T,0,\dots,0)\in(L[T]^+)^n$. So $\langle TX_1,X_2\rangle\neq 0$ and we have that $NK_2(A)\neq 0$.

Case II. If $k\subsetneqq k_1$ is an inseparable extension and A is K_2-regular we have by Corollary 2.1(ii) that A is K_1-regular which gives a contradiction with ([21], Lemma 4.4). So A cannot be K_2-regular.

Case III. We assume $\bar{A}=k_1[X]$ and $A=\{f\in k_1[X]|f(0)\in k\}$ where $k\subsetneqq k_1$ is a finite separable field extension.

Now it is possible to find a field k_2 such that $k\subsetneqq k_1\subset k_2$ and k_2 is a finite Galois extension of k. So $k_2=k(\alpha_1)$ where α_1 is a primitive element. If $f(Y)=Y^n+a_{n-1}Y^{n-1}+\dots+a_0$ is the monic minimum polynomial of α_1 over k we know that $n=[k_2:k]$ and $f(Y)=(Y-\alpha_1)\dots(Y-\alpha_n)$ splits into linear factors over k_2. Let $h(Y)$ be the monic minimum polynomial of α_1 over k_1. Then $f(Y)=g(Y)h(Y)$ with $\deg(h(Y))<n$.

Consider the following commutative diagram

$$\begin{array}{ccccc} NK_2(k_2\otimes_k A) & \xrightarrow{tr_1} & NK_2(A) & \xrightarrow{i_1} & NK_2(k_2\otimes_k A) \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} \\ NK_2(k_2\otimes_k(k+k_1\varepsilon)) & \xrightarrow{tr_2} & NK_2(k+k_1\varepsilon) & \xrightarrow{i_2} & NK_2(k_2\otimes_k(k+k_1\varepsilon)) \end{array}$$

The notation in this diagram will now be explained. First $k+k_1\varepsilon$ is the ring $A/(X^2k_1[X])$. One easily sees that $k_2\otimes_k A\cong A[Y]/f(Y)$ and that $k_2\otimes_k A$ is a free A-module. So ([18], §14) shows there exists a transfer map for K_2; hence by using polynomial extensions and taking kernels, there exists also a transfer for NK_2. We call this transfer map tr_1. Similar arguments give us a second transfer map tr_2. The other maps are all induced by natural morphisms of rings.

If we can find an element $\eta\in NK_2(k_2\otimes_k A)$ such that $(i_2\circ tr_2\circ j_1)(\eta)\neq 0$ we know that $tr_1(\eta)\neq 0$ in $NK_2(A)$ and we are done.

Let $h(Y)=Y^m+b_{m-1}Y^{m-1}+\dots+b_0$ and $g(Y)=Y^p+c_{p-1}Y^{p-1}+\dots+c_0$ with $b_i, c_i\in k_1$. Then we can consider for every $\omega\in k_2$ the elements $\left(\sum_{r=0}^{m}\omega\alpha_1^r\otimes b_rX\right)T$ and $\sum_{s=0}^{p}\alpha_1^s\otimes c_sX$ in $(k_2\otimes_k A)[T]$. We can calculate the product of these elements

$$\begin{aligned}\left(\sum_{r=0}^{m}\omega\alpha_1^r\otimes b_rX\right)T\cdot\left(\sum_{s=0}^{p}\alpha_1^s\otimes c_sX\right)&=\left(\sum_{t=0}^{n}\omega\alpha_1^t\otimes a_tX^2\right)T\\&=\left(\sum_{t=0}^{n}\omega a_t\alpha_1^t\otimes X^2\right)T=0.\end{aligned}$$

So the Dennis-Stein symbol

$$\left\langle\left(\sum_{r=0}^{m}\omega\alpha_1^r\otimes b_rX\right)T,\ \sum_{s=0}^{p}\alpha_1^s\otimes c_sX\right\rangle\in K_2((k_2\otimes_k A)[T])$$

is well-defined ([7], §1) and is even an element of $NK_2(k_2\otimes_k A)$.

Let

$$\eta_\omega=\left\langle\left(\sum_{r=0}^{m}\omega\alpha_1^r\otimes b_r\varepsilon\right)T,\ \sum_{s=0}^{p}\alpha_1^s\otimes c_s\varepsilon\right\rangle$$

be the image under j_1 of this element in $NK_2(k_2\otimes_k(k+k_1\varepsilon))$. Let $\sigma_1,\dots,\sigma_n$ be the k-automorphisms of k_2 such that $\sigma_i(\alpha_1)=\alpha_i$. By applying the calculations of Bloch (see [5], proof of Lemma 3.5.3) we see that $(i_2\circ tr_2)(\eta_\omega)=\prod_{i=1}^{n}\sigma_i(\eta_\omega)$ where

$$\sigma_i(\eta_\omega)=\left\langle\left(\sum_{r=0}^{m}\sigma_i(\omega\alpha_1^r)\otimes b_r\varepsilon\right)T,\ \sum_{s=0}^{p}\sigma_i(\alpha_1^s)\otimes c_s\varepsilon\right\rangle.$$

Furthermore $(k_2\otimes_k(k+k_1\varepsilon))[T]$ can be viewed as the ring with underlying abelian group $k_2[T]\oplus(k_2[T]\otimes_k k_1\varepsilon)$ and zero multiplication on the ideal $k_2[T]\otimes_k k_1\varepsilon$. So we have a natural split surjection

$$K_2(k_2[T]\oplus(k_2[T]\otimes_k k_1\varepsilon))\to K_2(k_2[T]).$$

Let $F_2(k_2[T]\otimes_k k_1\varepsilon)$ denote the kernel of this map. Dennis and Krusemeyer ([8], p. 6.4) describe a homomorphism of abelian groups

$$\theta : F_2(k_2[T]\otimes_k k_1\varepsilon)\to \Lambda^2(k_2[T]\otimes_k k_1\varepsilon)$$

(i.e. the second exterior power of the $k_2[T]$-module $k_2[T]\otimes_k k_1\varepsilon$). So it is enough to show that there exists an $\omega\in k_2$ such that $\theta\left(\prod_{i=1}^{n}\sigma_i(\eta_\omega)\right)\neq 0$. Now consider

$$\theta(\eta_1)=\left(\sum_{r=0}^{m}\alpha_1^r T\otimes b_r\varepsilon\right)\wedge\left(\sum_{s=0}^{p}\alpha_1^s\otimes c_s\varepsilon\right).$$

Assume that there exist $\lambda_1,\lambda_2\in k_2[T]$ such that

$$\lambda_1\left(\sum_{r=0}^{m}\alpha_1^r T\otimes b_r\varepsilon\right)+\lambda_2\left(\sum_{s=0}^{p}\alpha_1^s\otimes c_s\varepsilon\right)=0.$$

We have a $k_2[T]$-linear map $\phi : k_2[T]\otimes_k k_1\varepsilon\to k_2[T]$ with

$$\phi : f(T)\otimes v\varepsilon\mapsto vf(T).$$

So we have

$$\begin{aligned}0&=\phi\left(\lambda_1\left(\sum_{r=0}^{m}\alpha_1^r T\otimes b_r\varepsilon\right)+\lambda_2\left(\sum_{s=0}^{p}\alpha_1^s\otimes c_s\varepsilon\right)\right)\\&=\lambda_1 h(\alpha_1)T+\lambda_2 g(\alpha_1)=\lambda_2 g(\alpha_1).\end{aligned}$$

But $g(\alpha_1)\neq 0$, so $\lambda_2=0$. Hence we see that $\lambda_1=0$. We can conclude that $\theta(\eta_1)\neq 0$.

Let $e_1,\dots,e_s$ be a basis of k_1 over k. Then the $(1\otimes e_i\varepsilon)\wedge(1\otimes e_j\varepsilon)$ with $1\leqq i<j\leqq s$ form a basis of $\Lambda^2(k_2[T]\otimes_k k_1\varepsilon)$ over $k_2[T]$. Now we can order these $(1\otimes e_i\varepsilon)\wedge(1\otimes e_j\varepsilon)$, and call them f_l, such that $\theta(\eta_1)=\sum_{l=1}^{t}\varrho_l Tf_l$ with $\varrho_l\in k_2$ and $\varrho_1\neq 0$.

Now we have

$$\begin{aligned}\theta\left(\prod_{i=1}^{n}\sigma_i(\eta_\omega)\right)&=\sum_{i=1}^{n}\sum_{l=1}^{t}\sigma_i(\omega\varrho_l)Tf_l\\&=\sum_{l=1}^{t}\sum_{i=1}^{n}\sigma_i(\omega\varrho_l)Tf_l.\end{aligned}$$

But since the σ_i's are linearly independent we can find an $\omega\in k_2$ such that $\sum_{i=1}^{n}\sigma_i(\omega\varrho_1)\neq 0$. So for this $\omega\in k_2$ we have that $\theta\left(\prod_{i=1}^{n}\sigma_i(\eta_\omega)\right)\neq 0$. q.e.d.

3.5. Remark. Let $A=\{f\in k_1[X]\mid f_0\in k\}$ where $k\subsetneqq k_1$ is a separable field extension. We have shown that $NK_2(A)\neq 0$. But the Dennis-Stein symbols are not responsible for this result because one can easily show that all $\langle a,b\rangle\in K_2(A[T])$ are in fact elements of $K_2(A)$. So $K_2(A[T])$ contains more than only symbols.

We now come to the main theorem of this section.

3.6. Theorem. *Let k be a field. A a k-algebra essentially of finite type over k. Assume* $\dim A \leqq 1$. *Then the following conditions are equivalent.*

(i) *A is K_n-regular for some $n \geqq 2$.*

(ii) *A is regular.*

(iii) *A is K_n-regular for all $n \geqq 0$.*

Proof. (ii)$\Rightarrow$(iii) (see [19], p. 38), (iii)$\Rightarrow$(i) is trivial.

(i)$\Rightarrow$(ii). Let A be K_n regular for some $n \geqq 2$. By Corollary 2.1(ii) we have that A is K_2- and K_1-regular. So A is reduced. By Corollary 1.9 we may assume that A is local. But the assumptions say that $\bar{A}$ is a finite A-module and A is equicharacteristic. Then A is K_1-regular implies that A is seminormal by ([21], Theorem A). But then we can use Theorem 3.4 which says that A is regular. This finishes the proof.

Acknowledgements. The author is grateful to Dr. J. R. Strooker for his support and to Wilberd van der Kallen for inspiring discussions and especially for pointing out Corollary 2.1 and for writing the appendix.

References

1. Bak, A.: Strong approximation for central coverings of elementary groups. Preprint
2. Bass, H.: Algebraic K-theory. New York: Benjamin 1968
3. Bass, H.: Some problems in "classical" algebraic K-theory. Lecture Notes in Mathematics, 342, pp. 3–73. Berlin, Heidelberg, New York: Springer 1973
4. Bloch, S.: Algebraic K-theory and crystalline cohomology. Publ. Inst. des Hautes Études Scientifiques **47**, 187–268 (1978)
5. Bloch, S.: Some formulas pertaining to the K-theory of commutative groupschemes. Preprint
6. Davis, E.: On the geometric interpretation of seminormality. Proc. Amer. Math. Soc. **68**, 1–5 (1978)
7. Dennis, R.K., Stein, M.: K_2 of radical ideals and semi-local rings revisited. Lecture Notes in Mathematics, Vol. 342, pp. 281–303. Berlin, Heidelberg, New York: Springer 1973
8. Dennis, R.K., Krusemeyer, M.: $K_2(A[X, Y]/XY)$, a problem of Swan, and related computations. J. Pure Appl. Algebra **15**, 125–148 (1979)
9. Ferrand, D.: Les modules projectifs de type fini sur un anneau de polynomes sur un corps sont libres. Lecture Notes in Mathematics, 567, pp. 202–221. Berlin, Heidelberg, New York: Springer 1977
10. Godement, R.: Théorie des faisceaux. Paris: Hermann 1958
11. Grayson, D.: Higher algebraic K-theory II. Lecture Notes in Mathematics, 551, pp. 217–240. Berlin, Heidelberg, New York: Springer 1976
12. Grayson, D.: K-theory of endomorphisms. J. Algebra **48**, 439–446 (1977)
13. Grothendieck, A., Dieudonné, J.A.: Eléments de géométrie algébrique I. Berlin, Heidelberg, New York: Springer 1971
14. Grothendieck, A.: Eléments de géométrie algébrique III. Publ. Inst. Hautes Études Scientifiques **11** (1961)
15. Grothendieck, A.: Eléments de géométrie algébrique IV. Publ. Inst. Hautes Études Scientifiques **24** (1964)
16. Karoubi, M.: Localisation des formes quadratiques. Ann. Sci. École Norm. Sup. **4**, 359–404 (1974)
17. Maazen, H., Stienstra, J.: A presentation for K_2 of split radical pairs. J. Pure Appl. Algebra **10**, 271–294 (1978)
18. Milnor, J.: Introduction to algebraic K-theory. Annals of Mathematics Studies, Vol. 72. Princeton: University Press 1971
19. Quillen, D.: Higher algebraic K-theory I. Lecture Notes in Mathematics, 341, pp. 85–147. Berlin, Heidelberg, New York: Springer 1973
20. Stienstra, J.: Deformations of a second Chow group. Thesis, Utrecht (1978)
21. Vorst, T.: Polynomial extensions and excision for K_1. Math. Ann. (in press, 1979)
22. Weibel, C.A.: Homotopy in algebraic K-theory. Thesis, Chicago (1977)

Appendix

Wilberd van der Kallen

A1. Let R be an associative ring with unit, I a two-sided ideal in R, f a central element of R such that multiplication by f gives a bijection from I onto I. Then I can be identified with the ideal $R_f I$ of R_f.

Example. Take $R = A + XA_f[X]$, $I = XA_f[X]$ as in 1.3(∗).

A2. Let $K_2(R, I)$ denote the relative K-group which occurs in the long exact sequence

$$\ldots \to K_3(R) \to K_3(R/I) \to K_2(R, I) \to K_2(R) \to K_2(R/I) \ldots$$

We will use the characterization of $K_2(R, I)$ given by Keune and Loday. (See [2] and [3]. For definiteness we will use [2].) So $K_2(R, I)$ is the kernel of a map $\mathrm{St}(R, I) \to \mathrm{GL}(R)$, where $\mathrm{St}(R, I)$ is a certain relative Steinberg group. By giving another presentation of $\mathrm{St}(R, I)$, in the spirit of [1], we will be able to establish the following excision property.

A3. Theorem. *Let R, I, f be as in A1. Then the natural map $K_2(R, I) \to K_2(R_f, I)$ is an isomorphism.*

Remark. If the ring homomorphism $R \to R/I$ splits then $K_2(R, I)$ is simply the kernel of $K_2(R) \to K_2(R/I)$.

A4. Let A be an associative ring with unit, J a two-sided ideal in A, M a right A-module, N a left A-module, B a bilinear map $N \times M \to A$ (i.e. $B(n, m)$ is left A-linear in n and right A-linear in m). We will write nm for $B(n, m)$.

Definition. $\mathrm{St}(M, J, N)$ is the group defined by the following presentation.

Generators: $X(m, j, n)$ with $m \in M$, $j \in J$, $n \in N$, $nm = 0$.

Relations:

1. $X(m, aj, n) = X(ma, j, n)$
2. $X(m, ja, n) = X(m, j, an)$
3. $X(m, j_1 + j_2, n) = X(m, j_1, n) X(m, j_2, n)$
4. $X(m_1 + m_2, j, n) = X(m_1, j, n) X(m_2, j, n)$
5. $X(m, j, n_1 + n_2) = X(m, j, n_1) X(m, j, n_2)$
6. $X(v, i, w) X(m, j, n) X(v, i, w)^{-1} = X(m + viwm, j, n - nviw)$.

(We always assume that both sides are defined; e.g. one needs $m_i \in M$, $j \in J$, $n \in N$, $nm_i = 0$ in 4.)

A5. *Remarks.* (1) Note that we have suppressed A and B in the notation $\mathrm{St}(M, J, N)$.

(2) If $J = A$, everything can be expressed in terms of the $X(m, 1, n)$. But the rule $X(ma, 1, n) = X(m, 1, an)$ does only follow if $X(m, a, n)$ is defined, i.e. one needs $nm = 0$ and not just $anm = nma = 0$. So one has to take a little care when expressing the relations in terms of the $X(m, 1, n)$.

A6. Let $A^{(\infty)}$ denote the free two-sided A-module on the countable basis $e_1, e_2, \ldots$. We have a bilinear map $B: A^{(\infty)} \times A^{(\infty)} \to A$ given by the rule $B(e_r, e_s) = \delta_{rs}$. ($\delta_{rs} = 1$ if $r = s$; $\delta_{rs} = 0$ otherwise.)

Proposition. $\mathrm{St}(A) \simeq \mathrm{St}(A^{(\infty)}, A, A^{(\infty)})$ *via* $x_{rs}(a) \mapsto X(e_r, a, e_s)$.

Proof. This is easier than what we did in [1], as we are dealing with the stable case now. The inverse map $\mathrm{St}(A^{(\infty)}, A, A^{(\infty)}) \to \mathrm{St}(A)$ is defined as follows. If $m = \sum_{i=1}^{N} e_i a_i$, $n = \sum_{i=1}^{N} b_i e_i$, with $nm = 0$, choose $r, s > N$, $r \neq s$, and send $X(m, c, n)$ to $[x, [y, z]]$ where $x = \prod_{i=1}^{N} x_{is}(a_i)$, $y = x_{sr}(c)$, $z = \prod_{j=1}^{N} x_{rj}(b_j)$. As $nm = 0$ we have $[x, z] = 1$ and $[[x, y], [y, z^{-1}]] = 1$, from which it follows, by a formal computation, that $[x, [y, z]] = [[x, y], z]$. From this one easily sees that the image does not depend on r and not on s. It also follows that our prescription is consistent with relations 1 and 2 in A4. One checks the other relations as in [1], §3. (Or use [1] Theorem 2 with $n = \infty$.)

A7. Let J be a two-sided ideal in A.

Lemma. *If the ring homomorphism $A \to A/J$ splits, then* $\mathrm{St}(A, J) \simeq St(A^{(\infty)}, J, A^{(\infty)})$, *where* $\mathrm{St}(A, J)$ *is as in* [2].

Proof. Let $s : A/J \to A$ be the splitting homomorphism of rings. We let $\mathrm{St}(A/J)$ act on $\mathrm{St}(A^{(\infty)}, J, A^{(\infty)})$ as follows. The element $x_{pq}(\bar{a})$ acts by sending $X(m, j, n)$ to

$$X(m + e_p s(\bar{a}) e_q m, j, n - n e_p s(\bar{a}) e_q) = X(e_{pq}(s(\bar{a}))m, j, n e_{pq}(-s(\bar{a}))),$$

where in the right hand side we refer to the left and right actions, respectively, of $E(A)$ on $A^{(\infty)}$. Form the semi-direct product

$$\mathrm{St}(A^{(\infty)}, J, A^{(\infty)}) \rtimes \mathrm{St}(A/J)$$

and consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{St}(A, J) & \longrightarrow & \mathrm{St}(A) & \longrightarrow & \mathrm{St}(A/J) & \to & 1 \\
 & & & & {\scriptstyle\sigma}\downarrow \uparrow{\scriptstyle\varrho} & & \| & & \\
1 & \to & \mathrm{St}(A^{(\infty)}, J, A^{(\infty)}) & \to & \mathrm{St}(A^{(\infty)}, J, A^{(\infty)}) \rtimes \mathrm{St}(A/J) & \to & \mathrm{St}(A/J) & \to & 1
\end{array}$$

(The top row is exact because the homomorphism $A \to A/J$ splits.)

To define ϱ, send $\mathrm{St}(A^{(\infty)}, J, A^{(\infty)})$ into $\mathrm{St}(A^{(\infty)}, A, A^{(\infty)})$ in the natural way, send $\mathrm{St}(A/J)$ into $\mathrm{St}(A)$ using s, and apply Proposition A6.

To define σ, send $x_{ij}(a)$ to the product of $X(e_i, a - s(a/J), e_j)$ in $\mathrm{St}(A^{(\infty)}, J, A^{(\infty)})$ and $x_{ij}(a/J)$ in $\mathrm{St}(A/J)$. It is easy to see that ϱ and σ are inverse to each other.

A8. Theorem. $\mathrm{St}(A, J) \simeq \mathrm{St}(A^{(\infty)}, J, A^{(\infty)}) \simeq \mathrm{St}(A^{(\infty)}, A, JA^{(\infty)})$.

Proof. In [2] a description of $\mathrm{St}(A, J)$ is given using a group $\mathrm{Ker}(\mathrm{St}(A(J)_1) \xrightarrow[p_{1*}]{} \mathrm{St}(A))$, where p_{1*} is induced by a ring homomorphism $p_1 : A(J)_1 \to A$ which splits. Using Lemma A7 the first isomorphism in the theorem can be established by inspection of this description of Keune. Similarly one can find the second isomorphism by first proving an analogue of A7. Or one can prove the theorem using homomorphisms

$$\mathrm{St}(A, J) \to \mathrm{St}(A^{(\infty)}, J, A^{(\infty)}) \to \mathrm{St}(A^{(\infty)}, A, JA^{(\infty)}) \to \mathrm{St}(A, J)$$

and checking their compositions. Namely, recall that $\mathrm{St}(A, J)$ is a $\mathrm{St}(A)$-group with generators (as a $\mathrm{St}(A)$-group) which are called $y_{rs}(j)$. Send them to $X(e_r, j, e_s)$ and note that $\mathrm{St}(A^{(\infty)}, J, A^{(\infty)})$ is also a $\mathrm{St}(A)$-group via the rule

$$x_{rs}(a) \cdot X(m, j, n) = X(e_{rs}(a)m, j, ne_{rs}(-a)).$$

(Compare previous proof.)

One easily checks the necessary relations. (See [2], Theorem 12.) To go from $\mathrm{St}(A^{(\infty)}, J, A^{(\infty)})$ to $\mathrm{St}(A^{(\infty)}, A, JA^{(\infty)})$, simply send $X(m,j,n)$ to $X(m, 1, jn)$. Finally, to go from $\mathrm{St}(A^{(\infty)}, A, JA^{(\infty)})$ to $\mathrm{St}(A, J)$, send $X(m, c, n)$ to $[x, [y, z]]$ where $x = \prod_{i=1}^{N} x_{is}(a_i)$, $y = x_{sr}(c)$, $z = \prod_{j=1}^{N} y_{rj}(b_j)$, if $m = \sum_{i=1}^{N} e_i a_i$, $n = \sum_{j=1}^{N} b_j e_j$, and the commutators are to be computed in $\mathrm{St}(A, J) \rtimes \mathrm{St}(A)$. So $[x, [y, z]]$ is the same as

$$(x \cdot ((y \cdot z) z^{-1}))((y \cdot z) z^{-1})^{-1}.$$

Compare with A6.

A9. To prove Theorem A3 one checks that $\mathrm{St}(R, I) \cong \mathrm{St}(R_f, I)$ under the given conditions. The inverse of the map

$$\mathrm{St}(R^{(\infty)}, R, IR^{(\infty)}) \to \mathrm{St}(R_f^{(\infty)}, R_f, IR_f^{(\infty)})$$

is given by sending $X(mf^{-r}, af^{-s}, n)$ to $X(m, a, f^{-r-s}n)$ for $m \in R^{(\infty)}$, $a \in R$, $n \in IR^{(\infty)} = IR_f^{(\infty)}$. (One has to check, among other things, that if $mf^{-r} = \tilde{m}f^{-t}$, $af^{-s} = \tilde{a}f^{-u}$, one has $X(m, a, f^{-r-s}n) = X(\tilde{m}, \tilde{a}, f^{-t-u}n)$. Even if f is a zero divisor such checks are straightforward.)

References

1. Kallen, W.L.J. van der: Another presentation for Steinberg groups. Indag. Math. **39**, 304–312 (1977)
2. Keune, F.: The relativization of K_2. J. Algebra **54**, 159–177 (1978)
3. Loday, J.: Cohomologie et groupe de Steinberg relatifs. J. Algebra **54**, 178–202 (1978)

Received May 2, 1978