

Isomorphism classes of polarised abelian varieties and Drinfeld modules over finite fields

Valentijn Karemaker (Utrecht University)

Joint work with Bergström – Marseglia (IMRN, 2023)
and Katen – Papikian (J. Algebra, 2024)

Séminaire de Théorie Algorithmique des Nombres

11 June 2024

Abelian varieties over finite fields: set-up

Definitions

An **abelian variety** is a connected projective group variety.

The **dual variety** A^\vee of an abelian variety A over a field K is such that $A^\vee(\overline{K}) = \text{Pic}^0(A_{\overline{K}})$.

A **polarisation** of an abelian variety A is an isogeny $\mu : A \rightarrow A^\vee$ such that there exists an ample line bundle \mathcal{L} on $A_{\overline{K}}$ such that $\mu_{\overline{K}}$ equals $\varphi_{\mathcal{L}} : A \rightarrow A^\vee, x \mapsto [t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}]$.

When $K = \mathbb{F}_q$ is a finite field, abelian varieties over K are partitioned into **isogeny classes**.

Important open problem

Describe and compute (polarised!) isomorphism classes within a fixed polarised isogeny class.

Preliminaries: Complex multiplication

Definitions

A **CM-field** L/\mathbb{Q} is a totally imaginary quadratic extension L/L' of a totally real extension L'/\mathbb{Q} . It has a canonical involution $x \mapsto \bar{x}$.

A **CM-algebra** is a finite product of CM-fields.

A **CM-type** for a CM-algebra L is a subset $\Phi \subseteq \text{Hom}(L, \overline{\mathbb{Q}})$ so that

$$\text{Hom}(L, \overline{\mathbb{Q}}) = \Phi \amalg \overline{\Phi}.$$

An abelian variety A over K of dimension g **has CM (by (L, Φ))** if

$$L \subseteq \text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q}.$$

Fact

Every abelian variety over a finite field has CM.

Complex uniformisation

Consider an abelian variety A over \mathbb{C} of dimension g .

By **complex uniformisation**, we have

$$A(\mathbb{C}) \simeq \mathbb{C}^g / \Lambda, \quad \Lambda \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}.$$

When A has CM by (L, Φ) , we can say more:

There exists a fractional ideal I in L such that $A(\mathbb{C}) \simeq \mathbb{C}^g / \Phi(I)$.

Then also $A^\vee(\mathbb{C}) \simeq \mathbb{C}^g / \Phi(\bar{I}^t)$, where t is the trace dual. Hence,

$$\mathrm{Hom}_L(A, A^\vee) \leftrightarrow (\bar{I}^t : I) := \{x \in L : xI \subseteq \bar{I}^t\}.$$

Definition/construction

Let A be a g -dimensional abelian variety over a p -adic field K and with CM by (L, Φ) . Form $A_{\mathbb{C}} = A \otimes \mathbb{C}$; then $A_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^g / \Phi(I)$.

Write $\mathcal{H}(A) := I$. Then $\mathcal{H}(A^\vee) = \bar{I}^t$ and

$$\mathcal{H}(\mathrm{Hom}_L(A, A^\vee)) := \mathrm{Hom}_L(\mathcal{H}(A), \mathcal{H}(A^\vee)) = (\bar{I}^t : I).$$

Polarisations in characteristic zero

Let $\mathcal{H}(A) = I$, so $\mathcal{H}(A^\vee) = \bar{I}^t$ and $\mathcal{H}(\text{Hom}_L(A, A^\vee)) = (\bar{I}^t : I)$.

By definition, $\{ \text{polarisations of } A \} \subseteq \text{Hom}(A, A^\vee)$.

Proposition

Let A be a g -dimensional abelian variety over a p -adic field K and with CM by (L, Φ) . An L -linear isogeny $\mu : A \rightarrow A^\vee \in \text{Hom}(A, A^\vee)$ is a polarisation if and only if:

- $\mathcal{H}(\mu) = \lambda \in L$ is **totally imaginary** (i.e. $\bar{\lambda} = -\lambda$);
- λ is **Φ -positive** (i.e. $\text{Im}(\varphi(\lambda)) > 0$ for all $\varphi \in \Phi$).

(towards) Polarisations in characteristic p

Goal

Describe and compute polarisations of abelian varieties over finite fields $K = \mathbb{F}_q$.

Every A/\mathbb{F}_q has a Frobenius endomorphism π_A with characteristic polynomial $h_A(x) \in \mathbb{Z}[x]$, which is an isogeny invariant:

By Honda-Tate theory, $\{ \text{isogeny classes} \} \leftrightarrow \{ \text{char. poly's } h_A \}$.

Idea

Give analogous construction to \mathcal{H} for abelian varieties in characteristic p , to describe $\text{Hom}(A, A^\vee) \supseteq \{ \text{polarisations of } A \}$.

We will use the Centeleghe-Stix equivalence.

Categorical equivalence of Centeleghe-Stix

For this, we need to restrict to abelian varieties A_0 over \mathbb{F}_p such that h_{A_0} is squarefree ($\Leftrightarrow \text{End}(A_0)$ is commutative).

C-S equivalence

Fix an h as above, or equivalently an isogeny class AV_h .

Let $L := \mathbb{Q}[x]/(h) = \mathbb{Q}[F]$ and $V := p/F$.

Any $A_0 \in AV_h$ has $\text{End}(A_0) \supseteq \mathbb{Z}[F, V]$.

Choose $A_h \in AV_h$ with $\text{End}(A_h) = \mathbb{Z}[F, V]$.

Then the functor

$$\begin{aligned} \mathcal{G} : AV_h &\rightarrow \{ \text{fractional } \mathbb{Z}[F, V]\text{-ideals} \} \\ A_0 &\mapsto \text{Hom}(A_0, A_h), \text{ embedded into } L \end{aligned}$$

is an equivalence of categories.

Properties of the equivalence

We have the equivalence

$$\mathcal{G} : \text{AV}_h \rightarrow \{ \text{fractional } \mathbb{Z}[F, V]\text{-ideals} \}$$

$$A_0 \mapsto \text{Hom}(A_0, A_h), \text{ embedded into } L$$

There are some choices involved here:

- Choosing A_h : these form a $\text{Pic}(\mathbb{Z}[F, V])$ -orbit;
- Choosing an embedding into L .

Choosing well, we can ensure that $\mathcal{G}(A_0^\vee) = \overline{\mathcal{G}(A_0)}^t$ and hence

$$\mathcal{G}(\text{Hom}_L(A_0, A_0^\vee)) := (\mathcal{G}(A_0) : \mathcal{G}(A_0^\vee)) = (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t).$$

Compare: $\mathcal{H}(\text{Hom}(A, A^\vee)) = (\bar{I}^t : I)$.

Properties of the equivalence

We have the equivalence

$$\mathcal{G} : \text{AV}_h \rightarrow \{ \text{fractional } \mathbb{Z}[F, V]\text{-ideals} \}$$

$$A_0 \mapsto \text{Hom}(A_0, A_h), \text{ embedded into } L.$$

Assume that $\mathcal{G}(A_0^\vee) = \overline{\mathcal{G}(A_0)}^t$.

For $f : A_0 \rightarrow B_0$ and $f^\vee : B_0^\vee \rightarrow A_0^\vee$, we have $\mathcal{G}(f^\vee) = \overline{\mathcal{G}(f)}$. Also:

$$\begin{array}{ccc} \text{Hom}(B_0, B_0^\vee) & \xrightarrow{f^*} & \text{Hom}(A_0, A_0^\vee) \\ \downarrow \mathcal{G} & & \downarrow \mathcal{G} \\ (\mathcal{G}(B_0) : \overline{\mathcal{G}(B_0)}^t) & \xrightarrow{\mathcal{G}(f^*)} & (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t) \end{array}$$

where $f^* : \varphi \mapsto f^\vee \varphi f$, so $\mathcal{G}(f^*)$ is multiplication by $\mathcal{G}(f) \overline{\mathcal{G}(f)} \in L$.

Canonical liftings

Now $(\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t) = \mathcal{G}(\text{Hom}(A_0, A_0^\vee)) \supseteq \mathcal{G}(\text{polarisations})$.

Idea

Lift to characteristic zero to access the description of polarisations.
N.B.: $\text{Hom}(A_0, A_0^\vee)$ should be preserved by the lifting process.

Definition

A **canonical lifting** of A_0/\mathbb{F}_q to a local domain \mathcal{R} of characteristic zero with residue field \mathbb{F}_q and fraction field K is an abelian scheme \mathcal{A}/\mathcal{R} such that $\text{End}(A_0) = \text{End}(\mathcal{A})$ and $\mathcal{A} \otimes \mathbb{F}_q \simeq A_0$, $\mathcal{A} \otimes K \simeq A$.

N.B. : We may view $\text{End}(A_0)$ as an order in $L \simeq \text{End}^0(A_0)$; these identifications can be made compatibly with \mathcal{G} and \mathcal{H} .

Characteristic p versus characteristic zero

Proposition

If A_0/\mathbb{F}_q has a canonical lifting to A/K , or equivalently if A/K with CM by L has good reduction to A_0/\mathbb{F}_q , and if

$$\text{End}(A^\vee) \simeq \text{End}(A) \simeq \text{End}(A_0) \simeq \text{End}(A_0^\vee)$$

and it's Gorenstein, then reduction $\text{Hom}_L(A, A^\vee) \rightarrow \text{Hom}_L(A_0, A_0^\vee)$ is multiplication by some $\alpha \in \text{End}(A_0)^*$.

$$\begin{array}{ccccc}
 & & \text{Hom}(A_K, A_K^\vee) & & \\
 & & \downarrow \text{red} & \searrow \mathcal{H} & \\
 \text{Hom}(B_0, B_0^\vee) & \xrightarrow{f^*} & \text{Hom}(A_0, A_0^\vee) & & (\bar{I}^t : I) \\
 \downarrow \mathcal{G} & & \downarrow \mathcal{G} & & \downarrow \alpha \\
 (\mathcal{G}(B_0) : \mathcal{G}(B_0^\vee)) & \xrightarrow{\mathcal{G}(f)^*} & (\mathcal{G}(A_0) : \mathcal{G}(A_0^\vee)) & = & (\bar{I}^t : I)
 \end{array}$$

Main result: describing polarisations

Lemmas

- ① Let $f : A_0 \rightarrow B_0$ and $\mu_0 : B_0 \rightarrow B_0^\vee$ be isogenies. Then μ_0 is a polarisation $\Leftrightarrow f^* \mu_0 = f^\vee \mu_0 f$ is a polarisation.
- ② Let $\mu : A \rightarrow A^\vee$ be an isogeny and $\mu_0 : A_0 \rightarrow A_0^\vee$ its reduction. Then μ is a polarisation $\Leftrightarrow \mu_0$ is a polarisation.
- ③ The element $\alpha \in \text{End}(A) = \text{End}(A_0)$ is totally real: $\bar{\alpha} = \alpha$.

Theorem

Let h be a squarefree characteristic polynomial corresponding to the isogeny class AV_h over \mathbb{F}_p . Let $L \simeq \mathbb{Q}[x]/(h)$ and choose a CM-type Φ for L . Let $S = \bar{S}$ be a Gorenstein order in L such that there is an $A_0 \in AV_h$ with $\text{End}(A_0) = S$ which admits a canonical lifting to a p -adic field K . Then there exists a totally real $\alpha \in S^*$ such that for **any** $B_0 \in AV_h$ and any isogeny $\mu_0 : B_0 \rightarrow B_0^\vee$, μ_0 is a polarisation $\Leftrightarrow \alpha^{-1} \mathcal{G}(\mu_0) \in L$ is totally imaginary and Φ -positive.

When do canonical liftings exist?

Known results

- ① (Serre-Tate) Every **ordinary** AV has a canonical lifting.
- ② (Oswal-Shankar and BKM) Every **almost-ordinary** AV with commutative endomorphism ring has a canonical lifting.
- ③ (Bhatnagar-Fu) Certain abelian varieties with **real multiplication** have a canonical lifting.
- ④ (Chai-Conrad-Oort) Let h be irreducible, $L = \mathbb{Q}[x]/(h) = \mathbb{Q}[\pi]$ and Φ a CM-type such that (L, Φ) satisfies the **residual reflex condition (RRC)**.
Then the isogeny class corresponding to h contains an A_0/\mathbb{F}_q such that $\text{End}(A_0) = \mathcal{O}_L$ which has a canonical lifting.

- We generalised the RRC to squarefree h .
- Any AV separably isogenous to A_0 then also has a lifting.
- We implemented the (generalised) RRC in Magma.

Computation of polarisations

Under the assumptions of our theorem, there exists totally real $\alpha \in \mathcal{S}^*$ such that $\mu_0 : B_0 \rightarrow B_0^\vee$ is a polarisation if and only if $\alpha^{-1}\mathcal{G}(\mu_0) \in L$ is totally imaginary and Φ -positive.

To find all (principal) polarisations of B_0 starting with a given $\mathcal{G}(\mu_0) = i_0 \in L^*$, we need to compute

$\{i_0 u : u \in \text{End}(B_0)^* / \langle \nu \bar{\nu} \rangle \text{ s.t. } \alpha^{-1} i_0 u \text{ totally imaginary and } \Phi\text{-positive} \}$.

- $(B_0, \mu_0) \simeq (B_0, \mu'_0) \Leftrightarrow \exists \nu \in \text{End}(B_0)^* \text{ s.t. } \mathcal{G}(\mu_0) = \nu \bar{\nu} \mathcal{G}(\mu'_0)$.
- Can often ignore α ! E.g. if an AV with $\text{End} = \mathbb{Z}[F, V]$ lifts.

Aggregate examples

squarefree dimension 3			$p = 2$	$p = 3$	$p = 5$	$p = 7$
total			185	621	2863	7847
ordinary			82	390	2280	6700
almost ordinary			58	170	474	996
p -rank 1	no RRC		0	0	0	0
	yes RRC	5.5.2(R_w) yes	20	26	76	118
		5.5.2(R_w) no	4	16	12	8
p -rank 0	no RRC		0	3	2	1
	yes RRC	5.5.2(R_w) yes	20	15	17	23
		5.5.2(R_w) no	1	1	2	1

squarefree dimension 4			$p = 2$	$p = 3$
total			1431	10453
ordinary			656	6742
almost ordinary			392	2506
p -rank 2	no RRC		0	0
	yes RRC	5.5.2(R_w) yes	149	500
		5.5.2(R_w) no	49	312
p -rank 1	no RRC		6	36
	yes RRC	5.5.2(R_w) yes	80	184
		5.5.2(R_w) no	14	40
p -rank 0	no RRC		3	6
	yes RRC	5.5.2(R_w) yes	73	88
		5.5.2(R_w) no	9	39

Drinfeld modules over finite fields: set-up

We fix some notation:

- $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$.
- $\mathfrak{p} \trianglelefteq A$ is a prime of degree d , monic generator denoted by \mathfrak{p} .
- $k \cong \mathbb{F}_{q^n}$ is a finite extension of $A/\mathfrak{p} = \mathbb{F}_p \cong \mathbb{F}_{q^d}$.
- $\gamma: A \rightarrow A/\mathfrak{p} \hookrightarrow k$ is the A -field structure on k .

Let $\phi: A \rightarrow k\{\tau\}$ be a Drinfeld module over k of rank r , with $\mathcal{E} := \text{End}_k(\phi)$ and $D := \mathcal{E} \otimes_A F = \text{End}_k^0(\phi)$.

Let $\pi = \tau^n$ be the Frobenius endomorphism of k .

$$\begin{array}{ccc}
 & D & \\
 & | & \\
 & \tilde{F} = F(\pi) & \\
 & / \quad \backslash & \\
 F & & K = \mathbb{F}_q(\pi)
 \end{array}$$

We will consider the case where $D = \tilde{F}$ is commutative.

Guiding questions

The minimal polynomial of π over F determines an **isogeny class** of Drinfeld modules over k .

Important open problem

Describe, determine, and count the isomorphism classes within a fixed isogeny class.

- Brute force results for $r = 2, 3$. [Assong].
- Description of endomorphism rings due to Anglès, Garai-Papikian, Kuhn-Pink, and others.
- Related to calculating zeta functions of Drinfeld modular varieties.

Isogenies, subgroups, lattices, ideals [Laumon]

Let $u : \phi \rightarrow \psi$ be an isogeny of Drinfeld modules of rank r over k . The kernel of $u \in k\{\tau\}$ is a finite group scheme G_u in A -modules.

Let H_p denote the Dieudonné module and T_l the Tate module.

Via injective maps $u_p : H_p(\psi) \hookrightarrow H_p(\phi)$ and $u_l : T_l(\phi) \hookrightarrow T_l(\psi)$

for $l \neq p$, we find sublattices $M_p := u_p(H_p(\psi)) \subseteq H_p(\phi)$ and

$M_l := \text{Hom}(u_l^{-1} T_l(\psi), A_l) \subseteq \text{Hom}(T_l(\phi), A_l) =: H_l(\phi)$ for $l \neq p$,

and hence a sublattice $M := \prod_l M_l \subseteq \prod_l H_l(\phi) =: \mathbb{H}(\phi)$.

By construction, $G_u \simeq \prod_{l \neq p} H_l(\phi)/M_l \times H_p(\phi)/M_p = \mathbb{H}(\phi)/M$.

For an ideal $I \trianglelefteq \mathcal{E}$, we have $k\{\tau\}I = k\{\tau\}u_I$ for some $u_I \in k\{\tau\}$.

The sublattice corresponding to u_I is $I\mathbb{H}(\phi) = \prod_l IH_l(\phi)$, since

$\ker(u_I) = \phi[I] = \bigcap_{\alpha \in I} \ker(\alpha)$.

Ideal action on isomorphism classes [Hayes]

Recall $\phi : A \rightarrow k\{\tau\}$ is a Drinfeld module with $\mathcal{E} := \text{End}_k(\phi)$.
 For an ideal $I \trianglelefteq \mathcal{E}$, again write $k\{\tau\}I = k\{\tau\}u_I$ with $u_I \in k\{\tau\}$.

A Drinfeld module over k is determined by its value at T .
 Setting $\psi_T = u_I \phi_T u_I^{-1}$ determines a Drinfeld module ψ over k ,
 isogenous to ϕ via $u_I : \phi \rightarrow \psi$. We write $\psi = I * \phi$.

Lemma

The map $I \mapsto I * \phi$ determines an action of the monoid of fractional ideals of \mathcal{E} up to linear equivalence on the set of isomorphism classes in the isogeny class of ϕ whose endomorphism ring is the order of an \mathcal{E} -ideal (hence an overorder of \mathcal{E}).

When is this action free? When is it transitive?

Kernel ideals

Let $I \trianglelefteq \mathcal{E} := \text{End}_k(\phi) = D \cap k\{\tau\}$ be an ideal.

Definition

The ideal I is a **kernel ideal** if any of the following equivalent properties holds:

- 1 $I = (k\{\tau\}I) \cap D$. (Generally \subseteq .) [Yu]
- 2 $I = \text{Ann}_{\mathcal{E}}(\phi[I])$. (Generally \subseteq .)
- 3 For any $J \trianglelefteq \mathcal{E}$, we have $J\mathbb{H}(\phi) \subseteq I\mathbb{H}(\phi) \Rightarrow J \subseteq I$. (\Leftarrow holds.)

Lemma

Upon restricting to kernel ideals, the ideal action $I \mapsto I * \phi$ is free.

Lemma

Every ideal is a kernel ideal when \mathcal{E} is maximal, or when \mathcal{E} is Gorenstein, e.g., when $\mathcal{E} = A[\pi]$.

Endomorphism rings (under the ideal action)

Fix an isogeny class with commutative endomorphism algebra D .
The endomorphism ring \mathcal{E} of a Drinfeld module ϕ in the isogeny class is an order in D containing the minimal order $A[\pi]$.

For $I \trianglelefteq \mathcal{E}$, let $(I : I) = \{g \in D : Ig \subseteq I\}$ be its order.

Write $k\{\tau\}I = k\{\tau\}u_I$ as before.

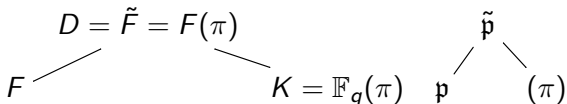
Lemma, cf. [Yu] and [Waterhouse]

For any $I \trianglelefteq \mathcal{E}$, we have $\text{End}_k(I * \phi) \supseteq u_I(I : I)u_I^{-1} \simeq (I : I)$.
Equality holds when I is a kernel ideal.

Since $\mathcal{E} \subseteq (I : I)$, “endomorphism rings grow under ideal action”.
For transitivity of $I \mapsto I * \phi$, every occurring endomorphism ring in the isogeny class should be an overorder of \mathcal{E} .

When does the minimal order $A[\pi]$ occur as endomorphism ring?

Local maximality of $A[\pi]$



Definition, cf. [Anglès]

Let $B_{\tilde{\mathfrak{p}}}$ be the ring of integers of $\tilde{F}_{\tilde{\mathfrak{p}}} := \tilde{F} \otimes_K \mathbb{F}_q((\pi))$ and write $A[\pi]_{\tilde{\mathfrak{p}}} := A[\pi] \otimes_{\mathbb{F}_q[[\pi]]} \mathbb{F}_q[[\pi]]$. Then $A[\pi]$ is **locally maximal** at π if $A[\pi]_{\tilde{\mathfrak{p}}} = B_{\tilde{\mathfrak{p}}}$.

Theorem

Recall $\deg(\mathfrak{p}) = d$ and $k \simeq \mathbb{F}_{q^n}$. Let H be the height of ϕ .

Then $\left\lceil \frac{n}{H \cdot d} \right\rceil \leq \frac{[\tilde{F}:K]}{d}$, with equality $\Leftrightarrow A[\pi]$ is locally maximal at π .
 Hence, $A[\pi]$ is locally maximal at $\pi \Leftrightarrow \phi$ is ordinary or $k = \mathbb{F}_p$.

$A[\pi]$ as an endomorphism ring

Fix an isogeny class with commutative endomorphism algebra D .

Lemma

Let R be any A -order in D containing π . There exists a Drinfeld module ϕ in the isogeny class such that $\text{End}_k(\phi) = R$ if and only if R is locally maximal at π .

At \mathfrak{p} , i.e. at π , any endomorphism ring is locally maximal. [Yu]

At all $\mathfrak{l} \neq \mathfrak{p}$, the order is almost always maximal and can be adjusted at the remaining places (\leftrightarrow isogeny).

Theorem: $A[\pi]$ is locally maximal at $\pi \Leftrightarrow \phi$ is ordinary or $k = \mathbb{F}_p$.

Corollary

$A[\pi]$ occurs as an endomorphism ring if and only if it is locally maximal at π , if and only if the isogeny class is ordinary or $k = \mathbb{F}_p$.
So does any overorder of $A[\pi]$.

Main result

Theorem

Suppose that $\mathcal{E} := \text{End}_k(\phi) = A[\pi]$. Then the action $I \mapsto I * \phi$ of the monoid of fractional ideals of $A[\pi]$ is free and transitive on the isomorphism classes in the isogeny class of ϕ .

In other words, all isomorphism classes in the isogeny class of ϕ are of the form $I * \phi$ for some $A[\pi]$ -ideal I .

- If $\mathcal{E} = A[\pi]$ then ϕ is ordinary or $k = \mathbb{F}_p$.
- For the Gorenstein order $A[\pi]$, every ideal is a kernel ideal.
- Kernel ideals act freely.
- Kernel ideals of $A[\pi]$ act transitively on isomorphism classes whose endomorphism ring is an overorder of $A[\pi]$, i.e. on all isomorphism classes.

Example

Let $q = 2$, $k = \mathbb{F}_4$, $\mathfrak{p} = T$. Fix $\alpha \in k \setminus \mathbb{F}_q$.

Let $\phi_1 : A \rightarrow k\{\tau\}$ be the (rank 7, height 1) Drinfeld module given by $(\phi_1)_T = \alpha\tau + \tau^2 + \tau^7$. Then $\text{End}_k(\phi_1) = A[\pi]$, $\pi = \tau^2$.

There are 15 isomorphism classes in the isogeny class of ϕ_1 :

I	u_I	$I * \phi_1$
(1)	1	ϕ_1
(T, π)	τ	ϕ_2
$(T^2 + T, \pi^3 + 1)$	$\alpha + \tau^3$	ϕ_3
$(T^2, \pi^2 + T + 1)$	$(\alpha + 1) + (\alpha + 1)\tau + \tau^3$	ϕ_4
$(T, \pi^4 + \pi^2 + \pi + 1)$	$1 + \alpha\tau^2 + \tau^3 + \tau^4$	ϕ_5
$(T + 1, \pi^3 + \pi + 1)$	$1 + (\alpha + 1)\tau + \tau^2 + \tau^3$	ϕ_6
$(T, \pi^2 + 1)$	$(\alpha + 1) + \tau + \tau^2$	ϕ_7
$(T^2 + T, \pi^3 + \pi^2 + \pi)$	$\tau + \alpha\tau^2 + \tau^3$	ϕ_8
$(T^2, \pi^2 + \pi + T)$	$(\alpha + 1)\tau + (\alpha + 1)\tau^2 + \tau^3$	ϕ_9
$(T, \pi^6 + \pi^5 + \pi^4 + \pi)$	$(\alpha + 1)\tau + \tau^2 + \alpha\tau^3 + \tau^4 + \alpha\tau^5 + \tau^6$	ϕ_{10}
$(T, \pi^3 + \pi^2 + 1)$	$(\alpha + 1) + \tau + \alpha\tau^2 + \tau^3$	ϕ_{11}
$(T^2, \pi + T + 1)$	$\alpha + \alpha\tau + \tau^2$	ϕ_{12}
$(T + 1, \pi^5 + \pi^4 + 1)$	$1 + \tau + (\alpha + 1)\tau^2 + (\alpha + 1)\tau^4 + \tau^5$	ϕ_{13}
$(T, \pi^4 + \pi^3 + \pi)$	$\alpha\tau + \tau^2 + (\alpha + 1)\tau^3 + \tau^4$	ϕ_{14}
$(T, \pi^2 + \pi)$	$(\alpha + 1)\tau + \tau^2$	ϕ_{15}

Comparing (polarised) abelian varieties and Drinfeld modules over finite fields k

In both cases we want to describe the isomorphism classes within a fixed isogeny class, determined by π .

We get the best results when the varieties/modules are **ordinary** or when k is the **prime field**.

Ordinary: canonical liftings exist; fractional End-ideals act on isomorphism classes – via ideal action (DM) or via complex uniformisation/Deligne's equivalence (AV).

Prime fields: elements with minimal endomorphism ring are key.
Centeleghe-Stix map $A_0 \mapsto \text{Hom}(A_0, A_h)$ with $\text{End}(A_h) = \mathbb{Z}[F, V]$.
Cf.: If $\phi = I * \phi_w$ with $\text{End}_k(\phi_w) = A[\pi]$ and I a kernel $A[\pi]$ -ideal, then $\text{Hom}_k(\phi, \phi_w) = I$.