# Polarisations of abelian varieties over finite fields via canonical liftings

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**Definition** An abelian variety is a non-singular connected projective group variety.

e.g. an elliptic curve

**Definition** The dual variety $A^\vee$ of an abelian variety $A$ over $K$ (↑ any field) is such that $A^\vee(\bar{K}) = \mathrm{Pic}^0(A_{\bar{K}})$.

e.g. for an elliptic curve $E$, $E^\vee \simeq E$.

**Definition** A polarisation of an abelian variety $A$ is an isogeny $\mu : A \to A^\vee$ such that there exists an ample line bundle $\mathcal{L}$ on $A_{\bar{K}}$ such that $\mu_{\bar{K}} = \varphi_{\mathcal{L}}$, where

$$\varphi_{\mathcal{L}} : A \to A^\vee$$
$$x \mapsto [t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}]$$

$\Rightarrow$ So $\{\text{polarisations of } A\} \subseteq \mathrm{Hom}(A, A^\vee)$.

$\boxed{\text{Goal}}$ Describe and compute polarisations of AV's when $K = \mathbb{F}_q$.

# § Preliminaries: Complex Multiplication

<u>Definition</u> A CM-field $L/\mathbb{Q}$ is such that

$$\begin{array}{ccl} & L & \\ \deg 2 & | & \text{tot. imaginary} \\ & L' & \\ \deg g & | & \text{totally real} \\ & \mathbb{Q} & \end{array}$$

It has a canonical involution $x \mapsto \bar{x}$.

A CM-algebra is a finite product of CM-fields.

<u>Definition</u> An abelian variety $A$ over $k$ of dimension $g$ has CM (by $L$) if $L \subseteq \mathrm{End}^\circ(A) := \mathrm{End}(A) \otimes \mathbb{Q}$.

<u>Fact</u> Every abelian variety over a finite field has CM.

<u>Definition</u> A CM-type for $L$ is a subset $\Phi \subseteq \mathrm{Hom}(L, \bar{\mathbb{Q}})$ such that $\mathrm{Hom}(L, \bar{\mathbb{Q}}) = \Phi \sqcup \bar{\Phi}$.

We often say an abelian variety "has CM by $(L, \Phi)$".

# § Polarisations in characteristic zero

Consider an abelian variety $A$ over a $p$-adic field $K$ of dimension $g$.

Complex uniformisation: $A_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda$, $\Lambda \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}$

When $A$ has CM by $(L, \Phi)$, we can say more:

$\exists$ fractional ideal $I$ in $L$ s.t. $A(\mathbb{C}) \simeq \mathbb{C}^g/\Phi(I)$. $\boxed{\mathcal{H}(A) := I}$.

Then also $A^\vee(\mathbb{C}) \simeq \mathbb{C}^g/\Phi(\overline{I}^t)$ ( $\overline{I}^t$ involution, trace dual )

i.e., $\mathcal{H}(A^\vee) = \overline{I}^t$,

and $\mathrm{Hom}_L(A, A^\vee) \overset{1:1}{\longleftrightarrow} (\overline{I}^t : I) = \{x \in L : xI \subseteq \overline{I}^t\}$

Recall: $\{\text{polarisations of } A\} \subseteq \mathrm{Hom}(A, A^\vee)$.

**Proposition** Let $A$ be a $g$-dimensional abelian variety over a $p$-adic field $K$ and with CM by $(L, \Phi)$.

An $L$-linear isogeny $\mu : A \to A^\vee \in \mathrm{Hom}(A, A^\vee)$ is a polarisation if and only if:

- $\mathcal{H}(\mu) = \lambda \in L$ is totally imaginary $(\overline{\lambda} = -\lambda)$
- $\lambda$ is $\Phi$-positive $(\mathrm{Im}(\varphi(\lambda)) > 0 \quad \forall \varphi \in \Phi)$

$$\boxed{\mathrm{Hom}(A, A^\vee) \xrightarrow{\ \mathcal{H}\ } (\overline{I}^t : I)}$$

# § (towards) polarisations in characteristic p

**Goal** Describe and compute polarisations of AV's when $K = \mathbb{F}_q$.
↳ as a subset $\{\text{polarisations of } A\} \subseteq \mathrm{Hom}(A, A^\vee)$

Every $A/\mathbb{F}_q$ has a Frobenius endomorphism $\pi_A$
which has a characteristic polynomial $h_A(x) \in \mathbb{Z}[x]$.

By Honda-Tate theory, $\{\text{isogeny classes}\} \xleftrightarrow{1:1} \{\text{charpoly's } h_A\}$

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**Centeleghe-Stix equivalence**

Fix such an $h \xleftrightarrow{HT}$ isogeny class $AV_h$

Let $L := \mathbb{Q}[x]/(h) = \mathbb{Q}[F]$ and $V := p/F$.

Any $A_0 \in AV_h$ has $\mathrm{End}(A_0) \supseteq \mathbb{Z}[F, V]$.

Choose $A_h \in AV_h$ with $\mathrm{End}(A_h) = \mathbb{Z}[F, V]$

Then $\forall A_0 \in AV_h$, $\mathcal{G} : AV_h \longrightarrow \{\text{fractional } \mathbb{Z}[F,V]\text{-ideals}\}$

$A_0 \longmapsto \mathrm{Hom}(A_0, A_h)$, embedded into $L$

N.B. For this, we need to restrict to:

Abelian varieties $A_0$ over $\mathbb{F}_p$ s.t. $h_{A_0}$ is squarefree.

We can ensure that $\mathcal{G}(A_0^\vee) = \overline{\mathcal{G}(A_0)}^t$ and hence

$$\mathcal{G}(\mathrm{Hom}_L(A_0, A_0^\vee)) := (\mathcal{G}(A_0) : \mathcal{G}(A_0^\vee)) = (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t)$$

[compare: $\mathcal{F}(\mathrm{Hom}(A, A^\vee)) = (\bar{I}^t : I)$]

$$\begin{tikzcd}
\mathrm{Hom}(B_0, B_0^\vee) \arrow[r, "f^*"] \arrow[d, "\mathcal{G}"'] & \mathrm{Hom}(A_0, A_0^\vee) \arrow[d, "\mathcal{G}"] & \\
(\mathcal{G}(B_0) : \mathcal{G}(B_0^\vee)) \arrow[r, "\mathcal{G}(f^*)"'] & (\mathcal{G}(A_0) : \mathcal{G}(A_0^\vee)) & = (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t)
\end{tikzcd}$$

# § Characteristic p versus characteristic zero

**Goal** Describe and compute polarisations of AV's when $K = \overline{\mathbb{F}}_p$

We now have $\mathcal{G}(\mathrm{Hom}(A_0, A_0^\vee)) \supset \mathcal{G}(\text{polarisations})$

$$\mathcal{G}(\mathrm{Hom}(A_0, A_0^\vee)) = (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t), \qquad \mathcal{G}(\text{polarisations}) = ???$$

**Idea** Lift to characteristic 0 to access description of polarisations.

N.B. $\mathrm{Hom}(A_0, A_0^\vee)$ should be preserved by the lifting process.

**Definition** A canonical lifting of $A_0/\mathbb{F}_q$ to a local domain $\mathcal{R}$ of characteristic 0 with residue field $\mathbb{F}_q$ and fraction field $K$ is an abelian scheme $\mathcal{A}/\mathcal{R}$ such that $\mathrm{End}(A_0) = \mathrm{End}(\mathcal{A})$ and

$$\mathcal{A} \otimes \mathbb{F}_q \simeq A_0, \qquad \mathcal{A} \otimes K \simeq A.$$

**Proposition** If $A_0/\mathbb{F}_q$ has a canonical lifting to $A/K$, and if $\mathrm{End}(A_0) \simeq \mathrm{End}(A_0^\vee)$ and $\mathrm{End}(A_0)$ is Gorenstein,

$$\begin{array}{ccc} \mathrm{End}(A_0) & \simeq & \mathrm{End}(A_0^\vee) \\ \text{is} & & \text{is} \\ \mathrm{End}(A) & \simeq & \mathrm{End}(A^\vee) \end{array}$$

then the reduction map $\mathrm{Hom}_L(A, A^\vee) \longrightarrow \mathrm{Hom}_L(A_0, A_0^\vee)$ is multiplication by $\alpha \in \mathrm{End}(A_0)^*$.

# § Characteristic $p$ versus characteristic zero

**Goal** Describe and compute polarisations of AV's when $K = \mathbb{F}_p$

Let $A$ be an abelian variety over a $p$-adic field $K$ with CM by $(L, \Phi)$ with good reduction to $A_0/\mathbb{F}_p \in AV_h$,
where $h$ is squarefree and has no real roots, and $L \simeq \mathbb{Q}[x]/(h)$,
such that $\mathrm{End}(A_0) = \mathrm{End}(A) = (I:I) = S$ is Gorenstein and satisfies $S = \bar{S}$.
Then we can make compatible choices to obtain

$$\begin{tikzcd}
 & \mathrm{Hom}(A, A^\vee) \arrow[dr, "\mathcal{H}"] \arrow[d, "\mathrm{red}", "\wr"'] & \\
\mathrm{Hom}(B_0, B_0^\vee) \arrow[r, "f^*"] \arrow[d, "\mathcal{G}"'] & \mathrm{Hom}(A_0, A_0^\vee) \arrow[d, "\mathcal{G}"] & (\bar{I}^t : I) \arrow[d, "\alpha", "\wr"'] \\
(\mathcal{G}(B_0) : \mathcal{G}(B_0^\vee)) \arrow[r, "\mathcal{G}(f^*)"'] & (\mathcal{G}(A_0) : \mathcal{G}(A_0^\vee)) & = (\mathcal{G}(A_0) : \overline{\mathcal{G}(A_0)}^t)
\end{tikzcd}$$

Also:

**Lemma** Let $f: A_0 \to B_0$ and $\mu_0: B_0 \to B_0^\vee$ be isogenies.
Then $\mu_0$ is a polarisation $\Leftrightarrow$ $f^*\mu_0 = f^\vee \circ \mu_0 \circ f$ is a polarisation

**Lemma** Let $\mu: A \to A^\vee$ be an isogeny and $\mu_0: A_0 \to A_0^\vee$ its reduction.
Then $\mu$ is a polarisation $\Leftrightarrow$ $\mu_0$ is a polarisation.

**Lemma** The element $\alpha \in S^*$ is totally real: $\alpha = \bar{\alpha}$.

# § Characteristic p versus characteristic zero

**Goal** Describe and compute polarisations of AV's when $K = \mathbb{F}_p$

Reformulating the above, we can now describe $\{\text{polarisations}\} \subseteq \mathrm{Hom}(A_0, A_0^\vee)$!

**Theorem** Let $h$ be a squarefree char. poly. without real roots corresponding to the isogeny class $AV_h$ over $\mathbb{F}_p$.
Let $L \simeq \mathbb{Q}[x]/(h)$ and choose a CM-type $\Phi$ for $L$.
Let $S = \overline{S}$ be a Gorenstein order in $L$ such that $\exists A_0 \in AV_h$ with $\mathrm{End}(A_0) = S$ which admits a canonical lifting to a $p$-adic field $K$.

Then there exists a totally real $\alpha \in S^*$ such that for any $B_0 \in AV_h$ and any isogeny $\mu_0 : B_0 \to B_0^\vee$,

$\mu_0$ is a polarisation $\iff \alpha^{-1}\mathcal{G}(\mu) \in L$ is totally imaginary and $\Phi$-positive.

$$\begin{tikzcd}
 & \mathrm{red}^{-1}(f^*\mu) \in \mathrm{Hom}(A,A^\vee) \arrow[d, "\mathrm{red}"] \arrow[dr, "\mathcal{H}"] & \\
\mu \in \mathrm{Hom}(B_0,B_0^\vee) \arrow[r, "f^*"] \arrow[d, "\mathcal{G}"'] & f^*\mu \in \mathrm{Hom}(A_0,A_0^\vee) \arrow[d, "\mathcal{G}"] & (\overline{I}^t : I) \ni \mathcal{H}(\mathrm{red}^{-1}(f^*\mu)) = \frac{1}{\alpha}\mathcal{G}(f^*\mu) \arrow[d, "\alpha"] \\
(\mathcal{G}(B_0):\mathcal{G}(B_0^\vee)) \arrow[r, "\mathcal{G}(f^*)"'] & (\mathcal{G}(A_0):\mathcal{G}(A_0^\vee)) \ni \mathcal{G}(f^*\mu) \arrow[r, phantom, "="] & (\overline{I}^t : I) \ni \mathcal{G}(f^*\mu)
\end{tikzcd}$$