Derived Torelli Theorem for K3 surfaces

Recap:

X projective variety, $/\mathbb{C}$.

$D^b(X) :=$ (bounded) derived category of coherent sheaves on X.

Obj: $$A^\bullet = \cdots \to A^0 \to A^1 \to A^2 \to \cdots$$

$$A_i \in Coh(X)$$

$$A \in Coh(X), \quad I^\bullet = I^0 \to I^1 \to \cdots$$

injective resolution of A

$$A = I^\bullet \in D^b(X)$$

- X, Y projective varieties, left exact functor:

$$F : Coh(X) \to Coh(Y)$$

induces a $\mathbf{R}F : D^b(X) \to D^b(Y)$

which is "exact" (preserves exact triangles).

Similar: F right exact $\Rightarrow$ $\mathbf{L}F$ "exact".

Fourier-Mukai Transform:

X, Y projective varieties. $P \in D^b(X \times Y)$.

$$\phi_P : D^b(Y) \longrightarrow D^b(X)$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

$$E \longmapsto Rp_*(Lq^*E \otimes^L P)$$

Derived Torelli:

Let X be a complex K3 surface. The integral cohomology:

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$
$$= H^2(X, \mathbb{Z}) \oplus U$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Mukai pairing:

$$\langle \alpha, \beta \rangle = (\alpha_2, \beta_2) - (\alpha_0, \beta_4) - (\alpha_4, \beta_0)$$

$\alpha = (\alpha_0, \alpha_2, \alpha_4)$
$\beta = (\beta_0, \beta_2, \beta_4)$

$(\cdot, \cdot)$ is the usual pairing.

$E, F \in D^b(X)$

$$\chi(E,F) = -\langle v(E), v(F) \rangle$$

- $\chi(\cdot,\cdot)$ Euler pairing
- $v(\cdot)$ Mukai vector.
- We put a weigth 2 Hodge structure on

$$(H^*(X,\mathbb{Z}), \langle \cdot,\cdot \rangle)$$

which we denote by $\widetilde{H}(X,\mathbb{Z})$:

$$\widetilde{H}(X,\mathbb{Z}) = \widetilde{H}^{2,0}(X) \oplus \widetilde{H}^{1,1}(X) \oplus \widetilde{H}^{0,2}(X)$$

$$\widetilde{H}^{2,0}(X) := H^{2,0}(X,\mathbb{Z})$$

$$\widetilde{H}^{1,1}(X) := H^{1,1}(X,\mathbb{Z}) \oplus \underbrace{H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})}_{\substack{\text{/\!/}2 \\ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}$$

- $\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{1,1}(X)$ with Mukai pairing.

Thm (Derived Torelli) [Mukai, Orlov]

Let X, Y be complex K3 surfaces. Then:

$$D^b(X) \xrightarrow{\simeq} D^b(Y) \iff \tilde{H}(X,\mathbb{Z}) \xrightarrow{\simeq} \tilde{H}(Y,\mathbb{Z})$$

Hodge isometry w.r.t. the Mukai pairing!

Cohomological FM:

$P \in D^b(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

$$\phi_P^H : \tilde{H}(X,\mathbb{Q}) \longrightarrow \tilde{H}(Y,\mathbb{Q})$$

$$\alpha \longmapsto p_*(q^*\alpha \cdot v(P))$$

Mukai vector:

$$v(P) := ch(P) \cdot \sqrt{td(X \times Y)}$$

$td(\cdot) = 1 + \dots$

Lemma

If $P \in D^b(X \times Y)$ defines an equivalence:

$$\phi_P : D^b(X) \xrightarrow{\simeq} D^b(Y)$$

Then: $\phi_P^H : H^*(X,\mathbb{Q}) \xrightarrow{\simeq} H^*(Y,\mathbb{Q})$

Proof:

Let ϕ_{P_R} be the adjoint of ϕ_P. Then:

- $\phi_{P_R} \circ \phi_P \cong id_{D^b(X)} \cong \phi_{O_{\Delta_X}}$
- $\phi_P \circ \phi_{P_R} \cong id_{D^b(Y)} \cong \phi_{O_{\Delta_Y}}$

- $\phi^H_{P_R} \circ \phi^H_P = \phi^H_{O_{\Delta_X}}$

  $\phi^H_P \circ \phi^H_{P_R} = \phi^H_{O_{\Delta_Y}}$

We need: $\phi^H_{\Delta_X} = id_X$

$$\begin{tikzcd} X \cong \Delta \arrow[r, hook, "i"] & X \times X \arrow[dl, "p"'] \arrow[d, "q"] \\ X & X \end{tikzcd}$$

$i_* O_\Delta$

$$v(O_\Delta) := ch(O_\Delta)\sqrt{td(X \times X)}$$

- $td(X) = i^* \sqrt{td(X \times X)}$

$$td(X \times X) = p^* td(X) \cdot q^* td(X)$$

$$i^* td(X \times X) = \underbrace{i^* p^*}_{id} td(X) \cdot \underbrace{i^* q^*}_{id} td(X)$$

$$= td(X)^2$$

GRR:

$$ch(i_* \mathcal{O}_\Delta) \cdot td(X \times X) = i_*\left(ch(\mathcal{O}_X) \cdot td(X)\right)$$

$$= i_*\left(1 \cdot i^* \sqrt{td(X \times X)}\right)$$

$$\overset{PF}{=} \sqrt{td(X \times X)} \cdot i_* 1$$

$$v(\mathcal{O}_\Delta) = i_* 1$$

$\beta \in H^*(X, \mathbb{Q})$

$$\phi^H_{\mathcal{O}_\Delta}(\beta) = q_*\left(p^*\beta \cdot v(\mathcal{O}_\Delta)\right)$$

$$= q_*\left(p^*\beta \cdot i_* 1\right)$$

$$PF = \underbrace{q_*(i_*}_{id}(\underbrace{i^* p^*}_{id}\beta))$$

$$p \circ i = id$$
$$q \circ i = id$$

$$= \beta.$$

$\square$

Prop: (Mukai, Căldăraru):

Let $\phi_P : D^b(X) \xrightarrow{\sim} D^b(Y)$, $P \in D^b(X \times Y)$.

Then $\phi_P^H : H^*(X, \mathbb{Q}) \xrightarrow{\sim} H^*(Y, \mathbb{Q})$.

induces:

(1) $$\bigoplus_{p-q=i} H^{p,q}(X) \cong \bigoplus_{p-q=i} H^{p,q}(Y)$$

(2) ϕ_P^H is isometric with respect to the Mukai pairing. $\square$

· X K3 surface:

$E \in D^b(X)$

$$v(E) := \left(\operatorname{rk} E,\ c_1(E),\ \frac{c_1^2(E)}{2} - c_2(E) + \operatorname{rk} E\right)$$

<u>Remark</u>:

$v(E)$ is an integral class!

$$\frac{c_1(E)^2}{2} - c_2(E) + \operatorname{rk}(E) = \chi(E) - \operatorname{rk}(E)$$

Hirzebruch-Riemann-Roch

More generally:

Lemma (Mukai):

X, Y K3 surfaces, $P \in D^b(X \times Y)$. Then:

$v(P) \in H^*(X \times Y, \mathbb{Q})$ is integral. $\square$

<u>Thm</u> (Derived Torelli):

X, Y K3 surfaces $/ \mathbb{C}$.

$$D^b(X) \simeq D^b(Y) \iff \widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(Y, \mathbb{Z})$$

<u>Proof</u>:

"$\Rightarrow$" $\quad \phi_P : D^b(X) \xrightarrow{\sim} D^b(Y)$, $\quad P \in D^b(X \times Y)$.

$$\Rightarrow \phi_P^H : H^*(X, \mathbb{Q}) \xrightarrow{\sim} H^*(Y, \mathbb{Q}).$$

$v(P) \in H^*(X \times Y, \mathbb{Z})$ so:

$$\phi_P^H : H^*(X, \mathbb{Z}) \xrightarrow{\sim} H^*(Y, \mathbb{Z}).$$

. H preserve the Hodge structure:

$$\bigoplus_{p-q=i} H^{p,q}(X) \cong \bigoplus_{p-q=i} H^{p,q}(Y)$$

$i=-2$: $\quad \tilde{H}^{0,2}(X) \cong \tilde{H}^{0,2}(Y)$

$i=0$: $\quad \underbrace{H^{1,1}(X) \oplus H^0(X) \oplus H^4(X)}_{\substack{\| \\ \tilde{H}^{1,1}(X)}} \cong \underbrace{H^{1,1}(Y) \oplus H^0(Y) \oplus H^4(Y)}_{\substack{\| \\ \tilde{H}^{1,1}(Y)}}$

$i=2$: $\quad \tilde{H}^{2,0}(X) \cong \tilde{H}^{2,0}(Y)$

· It preserves the Mukai pairing.

"$\Leftarrow$" Let $\varphi : \tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(Y, \mathbb{Z})$

be a Hodge isometry.

$$v := (r, \ell, s) = \varphi((0,0,1)) \in \tilde{H}(Y)$$

(i) $v = \pm(0,0,1)$.

$\Rightarrow \varphi(H^4(X)) = H^4(Y)$

$$(0,0,1)^{\perp} = H^2(X) \oplus H^4(X)$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H^0 \oplus H^4$$

$$\underset{\displaystyle (0,0,1)^{\perp}}{\overset{\displaystyle v^{\perp}}{\shortparallel}} \begin{matrix} = \varphi(H^2(X)) \oplus \varphi(H^4(X)) \\ = H^2(Y) \oplus H^4(Y) \end{matrix}$$

$\Rightarrow H^2(Y) \cong \varphi(H^2(X))$

$\varphi : H^2(X) \xrightarrow{\sim} H^2(\cancel{X})$

We use Global Torelli :

$$X \cong Y \implies D^b(X) \cong D^b(Y).$$

(ii) $v = (r, l, s) = \varphi(0,0,1)$ $\quad r \neq 0$.

Up to change of a sign on φ, suppose $r > 0$.

Consider the moduli space of stable sheaves on Y with Mukai vector v.

$$M := M_H(v)^s$$

Comes by computing $\mathrm{Ext}^1(E,E)$.

$$\dim M = 2 + \langle v, v \rangle$$
$$= 2 + \langle (0,0,1), (0,0,1) \rangle$$
$$= 2$$

General facts: if $\exists v'$ s.t. $\langle v, v' \rangle = 1$. Then $\exists$ universal family $\mathcal{E}$ on $Y \times M$. Take:

$v' = \varphi(-1, 0, 0) \implies \langle v, v' \rangle = 1$

- v is primitive, H generic $\implies M$ is projective.

$\Rightarrow M$ is K3 surface.

- $\mathcal{E} \in D^b(X \times Y)$ induces:

$$\phi_{\mathcal{E}} : D^b(M) \xrightarrow{\sim} D^b(Y)$$

$$\phi^H_{\mathcal{E}}((0,0,1)) = v$$

$$\tilde{H}(X) \xrightarrow[\sim]{\varphi} \tilde{H}(Y) \xrightarrow[\sim]{\phi^H_{\mathcal{E}}} \tilde{H}(M)$$

$$(0,0,1) \longmapsto v \longmapsto (0,0,1)$$

By (i) $\quad X \cong M \Rightarrow D^b(X) \cong D^b(M) \cong D^b(Y)$.

<u>Intermezzo</u>:

Spherical twists: $E \in D^b(Y)$ "cone"

$$\begin{array}{ccccccc}
E \boxtimes E^* & \longrightarrow & E \boxtimes E^*|_\Delta & \xrightarrow{tr} & \mathcal{O}_\Delta & \longrightarrow & P_E \\
\| & & \| & & \| & & \| \\
\mathcal{O}_{Y \times Y} & \longrightarrow & \mathcal{O}_\Delta & \xrightarrow{id} & \mathcal{O}_\Delta & \longrightarrow & \mathcal{O}(-\Delta)[1]
\end{array}$$

$E = \mathcal{O}_Y$

$$T_{\partial_\gamma}^H := \phi^H_{P_{\partial_\gamma}} = \phi^H_{\partial_\Delta} - \phi^H_{\partial_{\gamma\times\gamma}}$$

$$= id - \phi^H_{\partial_{\gamma\times\gamma}}$$

$$\begin{tikzcd} & Y \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y & & Y \end{tikzcd}$$

$$\nu(\partial_{Y\times Y}) = ch(\partial_{Y\times Y}) \cdot \sqrt{td(Y\times Y)}$$

$$= 1 \cdot p^*(1,0,1) \cdot q^*(1,0,1)$$

$$\phi^H_{\partial_{Y\times Y}} : H^*(Y, \mathbb{Z}) \longrightarrow H^*(Y, \mathbb{Z})$$

$$\alpha = (\alpha_0, \alpha_2, \alpha_4) \longmapsto q_*(p^*\alpha \cdot \nu(\partial_{Y\times Y}))$$

proj. formula

base change thm

$$\overset{*}{=} (\alpha_0 + \alpha_4, 0, \alpha_0 + \alpha_4)$$

$$T_{\partial_\gamma}^H = id - \phi^H_{\partial_{\gamma\times\gamma}}$$

$$= (-\alpha_4, \alpha_2, -\alpha_0)$$

(iii)

L line bundle

$$\tilde{H}(X,\mathbb{Z}) \xrightarrow[\sim]{\varphi} \tilde{H}(Y,\mathbb{Z}) \xrightarrow[\sim]{\cdot e^{c_1(L)} = ch(L)} \tilde{H}(Y,\mathbb{Z})$$

$$(0,0,1) \longmapsto (0,\ell,\underbrace{s}_{\text{possibly } 0}) = v \longmapsto (0,\ell,\underbrace{s+c_1(L)\cdot\ell}_{\text{for a suitable choice of } L})$$

$s + c_1(L)\cdot\ell \neq 0.$

$$* \quad \phi^H_{\Delta_* L} = e^{c_1(L)}$$

$$\Delta : Y \hookrightarrow Y \times Y$$

$$\tilde{H}(Y,\mathbb{Z}) \xrightarrow{T_{O_Y}} \tilde{H}(Y,\mathbb{Z})$$

$$(0,\ell,s+c_1(L)\cdot\ell) \longmapsto (\underbrace{-s+c_1(L)\cdot\ell}_{\neq 0},\ell,0)$$

$$(r',\ell',s')$$

We are in (ii) $\Rightarrow X \cong M$ moduli space over Y

$$\Rightarrow D^b(X) \cong D^b(M) \cong D^b(Y) \qquad \square$$

Remark:

If $D^b(X) \cong D^b(Y)$:

- $X \cong Y$ or
- $X \cong M_H(v)^s$ M moduli space of stable sheaves on Y.