Supersingular K3

Liedke:
lectures on
ss. K3.

SUPERSINGULAR K3s

1. Formal group laws
2. Artin supersingular
   vs.
   Shioda supersingular
3. Categorical equivalence
4. Unirational K3's.

1. FORMAL GROUP LAWS

Def: An n-dimensional formal group law over a field k consists of n power series $\vec{F} = (F_1, \dots, F_n)$

$$F_i = (x_1, \dots, x_n, y_1, \dots, y_n) \in k[[\vec{x}, \vec{y}]]$$

such that for all i holds:

- $F_i(\vec{x}, \vec{y}) \equiv x_i + y_i \quad (\text{mod } \deg \geq 2)$
- $F_i(\vec{x}, F_i(\vec{y}, \vec{z})) = F_i(F_i(\vec{x}, \vec{y}), \vec{z})$

F is called commutative if $F_i(\vec{x}, \vec{y}) = F_i(\vec{y}, \vec{x})$.

Examples: 1-dimensional

(a) Formal additive group: $\hat{G}_a = (F(x,y) = x + y)$.

(b) Formal multiplic. group: $\hat{G}_m = (F(x,y) = x + y + xy)$.

Def: A homomorphism $\vec{\alpha} : \vec{F} \to \vec{G}$, $\vec{F}$ n-dim. form. group law, G m-dim. form. group law consists of m formal power series in n variables

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \quad \text{such that:}$$

- $\alpha_i(\vec{x}) \equiv 0 \quad (\text{mod } \deg \geq 1)$
- $\vec{\alpha}(\vec{F}(\vec{x}, \vec{y})) = (\alpha_1(F_1, \dots, F_n), \dots, \alpha_m(F_1, \dots, F_n)) =$
  $= G(\alpha_1(\vec{x}), \dots, \alpha_m(\vec{x}), \alpha_1(\vec{y}), \dots, \alpha_m(\vec{y}))$

Example: For F commutative:

let $n \in \mathbb{Z}_{\geq 0}$:

$$[n](\vec{x}) := \underbrace{\vec{F}(\vec{F}(\ldots, \vec{x}), \vec{x})}_{n\text{-times}} \in k[[\vec{x}]].$$

$\boxed{\text{multiplication by } n.}$

Def: let $F = F(x,y)$ be a 1-dim form. group law. over k, with $\mathrm{char}(k) = p > 0$. Then the height $(h = ht(F))$ of F is defined to be:

(i) $h = \infty$. if $[p]F = 0$ $\boxed{F \text{ is unipotent}}$

(ii) $h = s$. if $\exists s \geq 1$ $[p]F = x^{p^s} + (\text{higher order terms})$ $\boxed{F \text{ is } p\text{-divisible}}$

Remark: If $\sigma: F \longrightarrow F^{(p)} = \sum\sum a_{ij}^p x_i y_j$ Frobenius map.

$z \longmapsto z^p$

then (i) is equivalent to σ being non-invertible.

(ii) is equivalent to σ being invertible.

FORMAL GROUP LAWS AND ALG. VAR.

X - smooth proper variety over k $\boxed{\text{Mazur and Artin}}$

$n \geq 1$ integer.

$\Phi^n_X : Art_k \longrightarrow$ Abelian groups.

$$A \longmapsto \ker\left(H^n_{et}(X \times_k A, \mathcal{O}^*_{X\times A}) \to H^n_{et}(X, \mathcal{O}^*_X)\right).$$

$\boxed{n=1}$ $\Phi^1_X = \{$ invertible sheaves $X \times_k A$ that trivialize when restricted to $X \}$.

$\Phi^1_X := \widehat{\operatorname{Pic}}X_k$. Formal Picard Group of X.

$\boxed{n=2}$

$\Phi^2_X : Art_k \longrightarrow$ Abelian groups.

$$A \longmapsto \ker\left(H^2_{et}(X \times_k A, \mathcal{O}^*_{X\times_k A}) \to \boxed{H^2_{et}(X, \mathcal{O}^*_X)}\right).$$

Brauer group of X.

In general Φ^2_X is not prorepres.

However, when X is a K3 $\underline{\text{it is}}$!!!

$$\Phi^2_X = \widehat{Br}\, X$$ Formal Brauer group of X.

$$\hat{\mu} : \hat{Br}X \times \hat{Br}X \longrightarrow \hat{Br}X .$$

$\rightsquigarrow$ equivalent to $\quad \hat{\mu}^{\#} : k[[\vec{x}]] \longrightarrow k[[\vec{y}, \vec{z}]]$

$\hat{\mu}$ is completely determined by the images of each x_i by $\hat{\mu}^{\#}$.

$\vec{G} = \hat{\mu}^{\#}(x_1, \ldots, x_n) \leftarrow$ n-dimensional formal group law.

2. SUPERSINGULARITY AND K3'S.

From now on $\mathrm{char}(k) = p > 0$.

Def: Let X a K3 surface, over k (a perfect field), let h be the height of its formal Brauer group.

We say X is Artin-supersingular iff $h = \infty$.

(*) Notice this is equivalent to the fact that the Frobenius is not invertible.

Def: Let X a K3, over k (a perfect field)

We say that X Shioda-supersingular iff $\rho(X) = 22$.

Remark: $\rho = 22 \Rightarrow h = \infty$ ($\Leftarrow$)

Shioda-supersingular $\Rightarrow$ Artin-supersingular

Tate's conjecture predicts the equivalence.

TATE'S CONJECTURE:

Def: Let k be a perfect field. Let $W = W(k)$ to be the Witt ring.

A K3 crystal is $(H, \varphi, \langle -,- \rangle)$.
$\hookrightarrow$ of rank n over k.

- H is a free W-module of rank n.
- $\varphi : H \longrightarrow H$ injective $\boxed{\sigma\text{-linear.}}$ $r \in W,\ m \in H$. $\varphi(rm) = \sigma(r)\varphi(m)$ $\sigma : W \to W$
- $\langle -,- \rangle : H \otimes_W H \longrightarrow H$ is a symmetric bilinear form.

(1) $p^2 H \subseteq \mathrm{Im}(\varphi)$.
(2) $\varphi \otimes_W k$ has rank 1.
(3) $\langle -,- \rangle$ is a perfect pairing
(4) $\langle \varphi(x), \varphi(y) \rangle = p^2 \sigma \langle x, y \rangle$.

Def: Let H a K3 crystal of rank n over k.
Then, we have $\mathbb{Z}_p$-module.

$$T_H = \{ x \in H : \varphi(x) = px \}.$$

This is what we call the Tate Module.

TATE'S CONJECTURE

If X a smooth proper alg. variety over $\mathbb{F}_q$ (alg. closed).
Then the following hold:

(1) we have (once we extend to the alg. closure)

$$NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\cong} T_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

(2) The rank of the $NS(X)$ is given by the orde of the pole $Z(X/\mathbb{F}_q, T)$ $T = q^{-1}$.

- Tate's[*] conjecture holds for abelian surfaces. [Tate], And also for products of curves.

- Tate's conjecure holds for K3 over odd characteristic. [Nygaard, Ogus, Charles, Madapusi Pera, Maulik].

$\boxed{\text{Theorem: Artin-supersingularity and Shioda-supersing. are equivalent notions for K3 over odd characteristic}}$

IDEA:

Let X a K3 over k, we can consider

- $H = H^2_{cris}(X|W)$.
- φ is the Frobenius on $H^2_{cris}(X|W)$.
- $\langle -,- \rangle$ - coming from Poincaré's duality.

$(H, \varphi, \langle -,- \rangle)$ is a K3 crystal of rank 22.

We can associate the Newton polygon to it.
Essentially it codifies the characteristic polynomial of the Frobenius.

~~R~~ then we say that H is supersingular if the Newton polygon is a straight line. $\Leftrightarrow$

$\Leftrightarrow$ the Frobenius is not invertible $\Leftrightarrow$

X is Artin s.s.

Proposition If $(H, \varphi, \langle -,- \rangle)$ a supersingular K3 crystal, ~~tet~~ and T_H is Tate module.

$$\mathrm{rank}_{\mathbb{Z}_p} T_H = \mathrm{rank}_W H.$$

$\langle -,- \rangle|_{T_H}$ induces a non-degenerate form, that is not perfect.

⊛ Assuming $X/\mathbb{F}_q$.

X Artin-ss. K3 $\Longrightarrow$ $H^2_{cris}(X/W)$ is a ss. K3-crystal

$$\Longrightarrow \mathrm{rank}_{\mathbb{Z}_p} T_H = \mathrm{rank}_W H = 22 \xRightarrow[\text{Tate's conjecture}]{}$$

$$\Longrightarrow \mathrm{rank}(NS(X)) = 22.$$

$\Longrightarrow$ X is Shioda-ss.

∎

EXAMPLES :

(a) Let A be a s.s. abelian surface in odd charact.

(A "is" isogenous to the product of ss. ell. curves.).

The $\mathrm{Kum}(A)$ is a supersingular K3 surface.

Thm: All ss. K3 admit an elliptic fibration.

(b). The Fermat quartic over $\mathbb{F}_q$ $(p \neq 2)$.

$$X_4 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \}.$$

X_4 is supersingular iff $p \equiv 3 \pmod 4$.

p odd
k alg. closed.

3. CATEGORICAL EQUIVALENCE (Groupoids)

What happens in the supersingular case?

Def: A supersingular K3 lattice is a free abelian group N of rank 22 together with an even symmetric bilinear form $\langle -,- \rangle$ s.t.

(1). The discriminant

$$d(N \otimes_{\mathbb{Z}} \mathbb{Q}) = -1 \in \mathbb{Q}^* / \mathbb{Q}^{*2}.$$

(2) The signature of

$N \otimes_{\mathbb{Z}} \mathbb{R}$ is $(1, 21)$.

(3) The cokernel $N \to N^\vee$ is killed by p. $\qquad \mathrm{Hom}(N, \mathbb{Z})$.

Def: $n \geq 1$, V is an $2n$-dimensional $\mathbb{F}_p$-vectorspace and $\langle -,- \rangle : V \times V \to \mathbb{F}_p$. a quadratic form.

non-degenerate and non-neutral
$\hookrightarrow \nexists$ n-dim isotropic subspace.

Let k be a perfect field of characteristic p.

$$\varphi = \mathrm{id}_V \otimes F_k : V \otimes_{\mathbb{F}_p} k \longrightarrow V \otimes_{\mathbb{F}_p} k$$

Frobenius on k.

We say that $K \subseteq V \otimes_{\mathbb{F}_p} k$ is characteristic

if.

(1). $K + \varphi(K)$ has dimension $n+1$

(2) K is a totally isotropic n-dim. subspace.

K is strictly characteristic

if $V \otimes_{\mathbb{F}_p} k = \sum_{i=1}^{\infty} \varphi^i(K)$ $= \varphi(\varphi(\cdots\varphi(K)))$

(*) In our case if $(N, \langle -,- \rangle)$ is a supersingular K3 lattice, we set $N_1 = N / pN^{\vee}$.

Then N_1 is a $(22-2\sigma_0)$-dimensional $\mathbb{F}_p$-vect space.

discriminant $(N) = -p^{2\sigma_0}$ $1 \leq \sigma_0 \leq 10$.

σ_0 = Artin invariant.

$K3(k) = \{$ category of ss. K3-crystals with only isomorph as morphisms$\}$

$\mathcal{O}3(k) = \{$ cat. of pairs (T, K) where

T is a ss. K3 lattice over $\mathbb{Z}_p$.

$K \subseteq {}^{pT^\vee}/_{pT} \otimes_{\mathbb{Z}} k$ is a strictly characteristic subspace

with only isomorph. as morphisms $\}$.

Theorem [Ogus]: There is an equivalence of groupoids.

$$K3(k) \longrightarrow \mathcal{O}3(k).$$

$$(H, \varphi, \langle -,- \rangle) \longmapsto (T_H, \underbrace{\boxed{\mathrm{Ker}\left(T_H \otimes_{\mathbb{Z}_p} k \to H \otimes_{\mathbb{Z}_p} k\right)}}_{K})$$

GEOMETRICAL IDEA:

Given H a ss. K3-crystal, we can find a ss. K3 X s.t. $H = H^2_{cris}(X/W)$.

If we consider T_H; this is a ss. K3-lattice.

And one can prove that K is a strictly characteristic subspace.

If T is ss. K3-lattice, we can construct

$$L = T \otimes W$$ $\boxed{\text{This is not a ss. K3-crystal!}}$

However, if K is a characteristic subspace

$$H = \underline{\varphi}^{-1}(\varphi(K)) \subseteq L.$$

$id \otimes F_k.$

$$\langle x, y \rangle_H = p^{-1} \langle x, y \rangle_L.$$

This is a ss. K3 crystal.

$$\overline{K} = \varphi(K) = \ker(T \otimes k \to H \otimes k).$$

If moreover, K is strictly characteristic

then : $T \cong T_H.$

which gives the equivalence

k alg closed.
positive char.
(Not ness. odd).

4. UNIRATIONAL K3 SURFACES

Def: An n-dimensional *unirational variety* as a variety X such that $k(X) \subseteq k(x_1, \dots, x_n)$.

We say that X is rational if $k(X) \cong k(x_1, \dots, x_n)$

Theorem [Shioda + ε]: Let X is a smooth proper unirational surface over k. then:

(i) $\rho(X) = 22$. (X is Shioda-supersingular).

$\Downarrow$

(ii) $H^2_{cris}(X/W)$ has as straight line as Newton Polygon. (X is Artin-supersingular)

$$\boxed{\text{Unirational} \Rightarrow \text{Shioda-ss.} \Rightarrow \text{Artin ss.}}$$

$\Leftrightarrow$ $\Leftrightarrow$

Conjecture: For K3 unirational iff supersingular

↳ Liedtk. $p \geq 5$ (2014).