Solution of Exercise 2.7. We consider the last case, of the mapping $f(x) = \langle x, x\rangle\, x$. For fixed $x \in \mathbf{R}^n$ and all $h \in \mathbf{R}^n$ we obtain

$$
\begin{aligned}
f(x+h) - f(x) &= \langle\, x+h, x+h\,\rangle(x+h) - \langle\, x, x\,\rangle\, x = (\langle\, x, x\,\rangle\, h + \langle\, x, h\,\rangle\, x + \langle\, h, x\,\rangle\, x) \\
&\quad +(\langle\, x, h\,\rangle\, h + \langle\, h, x\,\rangle\, h + \langle\, h, h\,\rangle\, x + \langle\, h, h\,\rangle\, h) =: Df(x)h + \epsilon_x(h).
\end{aligned}
$$

Here, with $l_x \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R})$ given by $l_x(h) = \langle\, x, h\,\rangle$, we have

$$
Df(x) = \|x\|^2 I + 2l_x\, x \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}^n).
$$

Furthermore, on the basis of the Cauchy–Schwarz inequality we see

$$
\|\epsilon_x(h)\| \le \|h\|^2(3\|x\| + \|h\|), \qquad \text{hence} \qquad \lim_{h\to 0} \frac{\|\epsilon_x(h)\|}{\|h\|} = 0.
$$

On the basis of Formula 2.10 we now see that the mapping f is differentiable on $\mathbf{R}^n$ with derivative $Df(x)$ at x as above.