Solution of Exercise 3.36. We introduce

$$F : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \qquad \text{by} \qquad F(x; a) = x - \int_x^a f(t)\,dt.$$

Then $F(0; b) = -\int_0^b f(t)\,dt = 0$. Furthermore, the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$ implies

$$\frac{d}{dx}\int_x^a f(t)\,dt = -f(x), \qquad \text{hence} \qquad D_x F(0; b) = 1 + f(0) \neq 0.$$

On the basis of the Implicit Function Theorem 3.5.1 there exists, for a in $\mathbf{R}$ sufficiently close to b, a unique solution $x = x(a) \in \mathbf{R}$ for $F(x(a); a) = 0$ with $x(a)$ near 0. Furthermore, the Fundamental Theorem implies that F is a C^1 function; therefore, it is a consequence of the Implicit Function Theorem that $a \mapsto x(a)$ is a C^1 function. We have

$$x'(b) = -D_x F(0; b)^{-1} D_a F(0; b) = \frac{f(b)}{1 + f(0)}.$$

Finally, suppose $f(t) = 2t - 1$ and $b = 1$. Then $1 + f(0) = 0$ and $\int_0^1 (2t - 1)\,dt = [t^2 - t]_0^1 = 0$; that is, the conditions of the Implicit Function Theorem are violated. Anyway, suppose there exists a solution $x = x(a) \in \mathbf{R}$ of $F(x; a)$ for every a in some neighborhood V of 1. Then we have, in particular, for $a \in V$ satisfying $0 < a < 1$,

$$x = \int_x^a (2t - 1)\,dt = [t^2 - t]_x^a = a^2 - a - x^2 + x, \qquad \text{that is} \qquad x^2 = a(a - 1) < 0.$$

This contradiction shows the absence in this case of a solution $x = x(a)$ defined for all a near 1.