Solution of Exercise 4.14.

(i) If we introduce the C^∞ function

$$g : \mathbf{R}^n \to \mathbf{R} \qquad \text{given by} \qquad g(x) = \langle Ax, x\rangle + \langle b, x\rangle + c,$$

then $V = g^{-1}(\{0\})$. According to the Submersion Theorem 4.5.2 we have that V is a C^∞ submanifold in $\mathbf{R}^n$ of dimension $n-1$ if g is a submersion at every point belonging to V. From Example 2.2.5 we obtain

$$Dg(x) \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}) \qquad \text{is given by} \qquad Dg(x) = 2Ax + b.$$

This mapping fails to be surjective only if it equals 0, that is, if $x = -\frac{1}{2}A^{-1}b$ in view of the invertibility of A. Such an x belongs to V only if

$$\frac{1}{4}\langle b, A^{-1}b\rangle - \frac{1}{2}\langle b, A^{-1}b\rangle + c = -\frac{1}{4}\langle b, A^{-1}b\rangle + c = 0, \qquad \text{i.e.} \qquad \Delta = 0.$$

(ii) This assertion follows immediately from the arguments in part (ii).

(iii) Using the fact that A is self-adjoint, we have

$$\begin{aligned} g(x) &= \langle A(y - \tfrac{1}{2}A^{-1}b), y - \tfrac{1}{2}A^{-1}b\rangle + \langle b, y - \tfrac{1}{2}A^{-1}b\rangle + c \\ &= \langle Ay - \tfrac{1}{2}b, y - \tfrac{1}{2}A^{-1}b\rangle + \langle b, y - \tfrac{1}{2}A^{-1}b\rangle + c \\ &= \langle Ay, y - \tfrac{1}{2}A^{-1}b\rangle + \tfrac{1}{2}\langle b, y - \tfrac{1}{2}A^{-1}b\rangle + c \\ &= \langle Ay, y\rangle - \tfrac{1}{2}\langle y, b\rangle + \tfrac{1}{2}\langle b, y\rangle - \tfrac{1}{4}\langle b, A^{-1}b\rangle + c = \langle Ay, y\rangle - \tfrac{1}{4}\Delta. \end{aligned}$$

Hence $x \in V$ if and only if $\langle Ay, y\rangle = \frac{1}{4}\Delta$. Since A is positive definite, we have $\langle Ay, y\rangle > 0$ according to Definition 2.9.2, for $y \in \mathbf{R}^n \setminus \{0\}$. Hence $\Delta < 0$ implies $V = \emptyset$. Furthermore, if $\Delta = 0$, then $y = 0$ corresponds to the only element $x \in V$; that is, $x = -\frac{1}{2}A^{-1}b$. Finally, if $\Delta > 0$, we obtain the equation that a quadratic form in y with positive eigenvalues equals a positive constant; and that is the equation of an ellipsoid.