Solution of Exercise 5.39.

(i) It is sufficient to prove the inequality under the extra assumption of $\|x\| = 1$; then it takes the form

$$n^n \prod_{1\le j\le n} x_j^2 \le 1.$$

Indeed, consider arbitrary $x \in \mathbf{R}^n \setminus \{0\}$; application of the special case to $\frac{1}{\|x\|}x$ then gives

$$n^n \prod_{1\le j\le n} \frac{x_j^2}{\|x\|^{2n}} \le 1.$$

Hence, we encounter the problem of showing that $f(x) = \prod_{1\le j\le n} x_j^2$ attains a maximum value equal to $\frac{1}{n^n}$ under the constraint $g(x) = \|x\|^2 - 1 = 0$. The method of Lagrange multipliers from Theorem 5.4.2 now leads to the system of equations, where $\lambda \in \mathbf{R}$,

$$2x_j \prod_{\substack{1\le i\le n\\ i\ne j}} x_i^2 = \lambda\, 2x_j \qquad (1 \le j \le n) \qquad \text{and} \qquad g(x) = 0.$$

This implies $x_j = 0$, for some $1 \le j \le n$; and then we find the minimum of f. The remaining case is that of $\prod_{1\le i\le n,\, i\ne j} x_i^2 = \lambda$; in other words, $\frac{f(x)}{x_j^2} = \lambda$, for all $1 \le j \le n$. This implies x_j^2 has a constant value, for all $1 \le j \le n$, which then must be equal to $\frac{1}{n}$ in view of $g(x) = 0$. But $f(x) = \frac{1}{n^n}$ under these conditions.

(ii) On the basis of the assumptions we may write $x_j^2 = \frac{a_j}{an}$; thus, $\|x\|^2 = \frac{1}{an}\sum_{1\le j\le n} a_j = 1$. In view of part (i) this gives

$$n^n \frac{\prod_{1\le j\le n} a_j}{a^n n^n} \le 1, \qquad \text{so} \qquad \prod_{1\le j\le n} a_j \le a^n, \qquad \text{hence} \qquad \Big(\prod_{1\le j\le n} a_j\Big)^{1/n} \le a = \frac{1}{n}\sum_{1\le j\le n} a_j.$$

(iii) Using the notation from Exercise 5.25.(i) we obtain by application of part (ii)

$$\sqrt[3]{(S-A_1)(S-A_2)(S-A_3)} \le \frac{1}{3}\Big(3S - \sum_{1\le j\le 3} A_j\Big) = \frac{S}{3},$$

with equality if all sides are of equal length. On the basis of the same exercise we now find

$$O^2 = S(S-A_1)(S-A_2)(S-A_3) \le S\frac{S^3}{3^3} = \frac{l^4}{2^4\cdot 3^3}, \qquad \text{so} \qquad O \le \frac{l^2}{12\sqrt{3}}.$$