Solution of Exercise 5.6.

(i) If we introduce the C^∞ function

$$g : \mathbf{R}^n \to \mathbf{R} \qquad \text{given by} \qquad g(x) = \langle Ax, x\rangle + \langle b, x\rangle + c,$$

then $V = g^{-1}(\{0\})$. From Exercise 4.14.(i) and (ii) we know that W is a C^∞ submanifold of $\mathbf{R}^n$ of dimension $n-1$. And from Example 2.2.5 we obtain

$$Dg(x) \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}) \qquad \text{is given by} \qquad Dg(x) = 2Ax + b.$$

Therefore the equation for T_xV follows from Theorem 5.1.2.

(ii) The result in part (i) immediately gives $x + T_xV = \{\, h \in \mathbf{R}^n \mid \langle 2Ax + b, h\rangle = \langle 2Ax + b, x\rangle \,\}$ and this implies the first equation. Expansion of the first equation leads to

$$2(\langle Ax, x\rangle + \langle b, x\rangle + c) - \langle b, x\rangle - 2c - \langle 2Ax + b, h\rangle = 0.$$

The second formula now follows by using that $g(x) = 0$ and multiplying by $-\frac{1}{2}$.