Solution of Exercise 6.49.

(i) Introducing polar coordinates $x = r(\cos\alpha, \sin\alpha)$ in $\mathbf{R}^2$ as in Example 6.6.4, we get $|x_1 + ix_2| = r|e^{i\alpha}| = r$ and $dx = r\,dr\,d\alpha$; therefore we have by the same example

$$\begin{aligned}&\int_{\mathbf{R}^2} \frac{1}{|x_1 + i\,x_2|}\frac{1}{2}|D_1 f(x) + iD_2 f(x)|\,dx\\ &\quad\le \int_{-\pi}^{\pi}\int_{\mathbf{R}_+} (|D_1 f(r\cos\alpha, r\sin\alpha)| + |D_2 f(r\cos\alpha, r\sin\alpha)|)\,dr\,d\alpha < \infty,\end{aligned}$$

in view of the compactness of the support of f.

(ii) The desired equality follows from the following identities:

$$\frac{1}{x_1 + i\,x_2} = \frac{e^{-i\alpha}}{r}, \qquad dx = r\,dr\,d\alpha,$$

$$D_1 f(x) = \cos\alpha\frac{\partial\widetilde{f}}{\partial r}(r,\alpha) - \frac{\sin\alpha}{r}\frac{\partial\widetilde{f}}{\partial\alpha}(r,\alpha), \qquad D_2 f(x) = \sin\alpha\frac{\partial\widetilde{f}}{\partial r}(r,\alpha) + \frac{\cos\alpha}{r}\frac{\partial\widetilde{f}}{\partial\alpha}(r,\alpha),$$

$$D_1 f(x) + iD_2 f(x) = e^{i\alpha}\left(\frac{\partial\widetilde{f}}{\partial r} + \frac{i}{r}\frac{\partial\widetilde{f}}{\partial\alpha}\right)(r,\alpha).$$

(iii) In view of Example 6.6.4 once more, the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$ and the compactness of the support of f successively, we find

$$\int_{\mathbf{R}_+}\int_{-\pi}^{\pi}\frac{\partial\widetilde{f}}{\partial r}(r,\alpha)\,d\alpha\,dr = \int_{-\pi}^{\pi}\int_{\mathbf{R}_+}\frac{\partial\widetilde{f}}{\partial r}(r,\alpha)\,dr\,d\alpha = \int_{-\pi}^{\pi} -f(0)\,d\alpha = -2\pi f(0).$$

(iv) Obviously the function $\alpha \mapsto \widetilde{f}(r,\alpha) = f(r\cos\alpha, r\sin\alpha)$ is 2π-periodic. As a consequence $(\star)$ follows from part (iii) and

$$\int_{-\pi}^{\pi}\frac{\partial\widetilde{f}}{\partial\alpha}(r,\alpha)\,d\alpha = \widetilde{f}(r,\pi) - \widetilde{f}(r,-\pi) = 0.$$