Solution of Exercise 6.59. In view of Corollary 6.4.3 on writing an iterated integral as a two-dimensional integral we obtain

$$\Gamma(x)\Gamma(1-x) = \int_{\mathbf{R}_+} e^{-t_1} t_1^{x-1}\, dt_1 \int_{\mathbf{R}_+} e^{-t_2} t_2^{-x}\, dt_2 = \int_{\mathbf{R}_+^2} e^{-(t_1+t_2)} \Big(\frac{t_1}{t_2}\Big)^x \frac{1}{t_1}\, dt.$$

According to Exercise 3.1 we have the diffeomorphism $\Psi : \mathbf{R}_+^2 \to \mathbf{R}_+^2$ satisfying

$$s = \Psi(t) = (t_1 + t_2, \frac{t_1}{t_2}) \qquad \text{and} \qquad ds = |\det D\Psi(t)|\, dt = \frac{t_1 + t_2}{t_2^2}\, dt.$$

Application of the Change of Variables Theorem 6.6.1 and Corollary 6.4.3 once more now gives

$$\begin{aligned}\Gamma(x)\Gamma(1-x) &= \int_{\mathbf{R}_+^2} e^{-(t_1+t_2)} \Big(\frac{t_1}{t_2}\Big)^{x-1} \frac{1}{\frac{t_1}{t_2}+1} \frac{t_1+t_2}{t_2^2}\, dt = \int_{\mathbf{R}_+^2} e^{-s_1} \frac{s_2^{x-1}}{s_2+1}\, ds = \int_{\mathbf{R}_+} \frac{s_2^{x-1}}{s_2+1}\, ds_2 \\ &= \frac{\pi}{\sin \pi x},\end{aligned}$$

where the last equality follows from Exercise 0.15.(iv).