Solution of Exercise 7.15.

(i) Note

$$x \in V_1 \cap V_2 \qquad \Longleftrightarrow \qquad \Big(x_1^2 + x_2^2 + x_3^2 = 1 \qquad \text{and} \qquad (x_1 - \frac{1}{2})^2 + x_2^2 = \frac{1}{4}\Big).$$

The former equation gives the existence of $\alpha \in \,]-\pi, \pi\,]$ such that $x_1^2 + x_2^2 = \cos^2\alpha$ and $x_3 = \sin\alpha$. Subtraction of $x_1^2 + x_2^2 = \cos^2\alpha$ and $(x_1 - \frac{1}{2})^2 + x_2^2 = \frac{1}{4}$ implies

$$x_1 - \frac{1}{4} = \cos^2\alpha - \frac{1}{4}, \qquad \text{so} \qquad x_1 = \cos^2\alpha \qquad \text{and} \qquad x_2^2 = \cos^2\alpha - \cos^4\alpha = \cos^2\alpha\sin^2\alpha.$$

(ii) Obviously, ϕ is a C^∞ mapping. First we show that the restrictions of ϕ are injective. Indeed, suppose $\phi(\alpha) = \phi(\widetilde{\alpha})$. Consideration of the last two coordinates then gives $\sin\alpha = \sin\widetilde{\alpha}$ and $\cos\alpha = \cos\widetilde{\alpha}$, that is, $\alpha = \widetilde{\alpha}$, unless $\sin\alpha = \sin\widetilde{\alpha} = 0$. But the latter condition means $\alpha \in \{-\pi, 0, \pi\}$. Next, we have

$$D\phi(\alpha) = (-2\cos\alpha\sin\alpha, -\sin^2\alpha + \cos^2\alpha, \cos\alpha) = (-\sin 2\alpha, \cos 2\alpha, \cos\alpha),$$

which proves that ϕ is an immersion everywhere. Finally, the continuity of ϕ^{-1} restricted to its image follows as in Example 3.1.1. Hence the restrictions of ϕ satisfy the conditions of being an embedding as given in Definition 4.2.4.

(iii) Part (ii) implies

$$\|D\phi(\alpha)\|^2 = 1 + \cos^2\alpha = 2 - \sin^2\alpha = 2(1 - \frac{1}{2}\sin^2\alpha).$$

Now note that the right–hand side is invariant under $\alpha \mapsto -\alpha$ and $\alpha \mapsto \pi - \alpha$. According to Formula (7.16) the length of $V_1 \cap V_2$ equals

$$4\int_0^{\frac{\pi}{2}} \|D\phi(\alpha)\|\, d\alpha.$$

(iv) The equality of the orthogonal projection of V_1 to D is obvious, while the second equality for D follows from the following. The boundary of D equals the circle $(y_1 - \frac{1}{2})^2 + y_2^2 = \frac{1}{4}$, which allows the parametrization

$$y_1 = \frac{1}{2} + \frac{1}{2}\cos\beta = \frac{1}{2}(1 + \cos\beta) = \cos^2\frac{\beta}{2}, \qquad y_2 = \frac{1}{2}\sin\beta = \cos\frac{\beta}{2}\sin\frac{\beta}{2},$$

where $-\pi < \beta \le \pi$. The desired formula now is obtained by writing $\frac{\beta}{2} = \alpha$. Note that the parametrization $\alpha \mapsto \cos\alpha(\cos\alpha, \sin\alpha)$ of the boundary of D coincides with the projection of $\phi(\alpha)$ onto the plane $\{\, x \in \mathbf{R}^3 \mid x_3 = 0 \,\}$. Furthermore, the description of L follows from the equality $L = (D \times \mathbf{R}) \cap \{\, (y, x_3) \in \mathbf{R}^3 \mid |x_3| \le \sqrt{1 - \|y\|^2} \,\}$, since L is the intersection of the solid cylinder and the unit ball.

(v) On the basis of Theorem 6.4.5 the description of L in part (iv) immediately leads to

$$\begin{aligned} \mathrm{vol}_3(L) &= 2\int_D \sqrt{1 - \|y\|^2}\, dy = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\alpha} r\sqrt{1 - r^2}\, dr\, d\alpha \\ &= 4\int_0^{\frac{\pi}{2}} \Big[-\frac{1}{3}(1 - r^2)^{\frac{3}{2}}\Big]_0^{\cos\alpha} d\alpha = \frac{4}{3}\int_0^{\frac{\pi}{2}} (1 - \sin^3\alpha)\, d\alpha = \frac{2}{3}\pi - \frac{4}{3}\,\frac{2}{3}. \end{aligned}$$

Here, the second equality follows by introducing polar coordinates in D and the hint has been used in the last equality.

(vi) As in Example 7.4.10 one sees

$$\begin{aligned}
\operatorname{area}_2(V_1) &= 2\int_{\text{projection}(V_1)} \frac{1}{\sqrt{1-\|y\|^2}}\,dy = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\alpha} \frac{r}{\sqrt{1-r^2}}\,dr\,d\alpha \\
&= 4\int_0^{\frac{\pi}{2}} \Big[-\sqrt{1-r^2}\Big]_0^{\cos\alpha}\,d\alpha = 4\int_0^{\frac{\pi}{2}} (1-\sin\alpha)\,d\alpha = 2\pi + 4\big[\cos\alpha\big]_0^{\frac{\pi}{2}} \\
&= 2\pi - 4.
\end{aligned}$$

(vii) From the parametrization of $V_1 \cap V_2$ in part (i) and that of the boundary of D as given in part (iv) we see that $V_2 \cap \mathbf{R}^3_+$ is parametrized by

$$\chi : (\alpha, x_3) \mapsto \begin{pmatrix} \cos^2\alpha \\ \cos\alpha\sin\alpha \\ x_3 \end{pmatrix} \qquad (0 < \alpha < \frac{\pi}{2},\ 0 < x_3 < \sin\alpha).$$

Accordingly, with the partial derivatives being evaluated at (α, x_3),

$$\frac{\partial\chi}{\partial\alpha} = \begin{pmatrix} -2\sin\alpha\cos\alpha \\ \cos^2\alpha - \sin^2\alpha \\ 0 \end{pmatrix}, \qquad \frac{\partial\chi}{\partial x_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \frac{\partial\chi}{\partial\alpha} \times \frac{\partial\chi}{\partial x_3} = \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}.$$

In particular, $\big\|\frac{\partial\chi}{\partial\alpha} \times \frac{\partial\chi}{\partial x_3}(\alpha, x_3)\big\| = 1$. Furthermore, V_2 being invariant under the reflections $x_j \mapsto -x_j$, for $j \in \{2,3\}$, we have by symmetry that $\operatorname{area}_2(V_2) = 4\operatorname{area}_2(V_2 \cap \mathbf{R}^3_+)$; and so

$$\operatorname{area}_2(V_2) = 4\int_0^{\frac{\pi}{2}}\int_0^{\sin\alpha} dx_3\,d\alpha = 4\int_0^{\frac{\pi}{2}} \sin\alpha\,d\alpha = 4\big[-\cos\alpha\big]_0^{\frac{\pi}{2}} = 4.$$