Solution of Exercise 7.17.

(i) We have

$$D\phi(s,t) = \begin{pmatrix} t\gamma'(s) & \gamma(s) \end{pmatrix} \in \mathrm{Lin}(\mathbf{R}^2, \mathbf{R}^3).$$

This is a matrix of rank 2, for all $(s,t) \in D$, since $\gamma(s)$ and $t\gamma'(s)$ are linearly independent in this case. Therefore the kernel of $D\phi(s,t)$ is of dimension 0 and thus $D\phi(s,t)$ is injective. It follows that ϕ is a C^1 immersion.

(ii) Using the symbol t with two different meanings, we obtain the following element in $\mathrm{Mat}(2,\mathbf{R})$:

$$D\phi(s,t)^t\, D\phi(s,t) = \begin{pmatrix} t\gamma'(s) \\ \gamma(s) \end{pmatrix} \begin{pmatrix} t\gamma'(s) & \gamma(s) \end{pmatrix} = \begin{pmatrix} t^2\langle \gamma'(s), \gamma'(s)\rangle & t\langle \gamma'(s), \gamma(s)\rangle \\ t\langle \gamma(s), \gamma'(s)\rangle & \langle \gamma(s), \gamma(s)\rangle \end{pmatrix}.$$

On the basis of Formula (5.3) in the case of $n = 3$ this implies

$$\det\left(D\phi(s,t)^t\, D\phi(s,t)\right) = t^2 \|\gamma'(s) \times \gamma(s)\|^2.$$

It follows that

$$\begin{aligned} \mathrm{area}(C) &= \int_{I\times\,]\,0,1\,[} t\|\gamma(s) \times \gamma'(s)\|\, d(s,t) = \int_0^1 t\, dt \int_I \|\gamma(s) \times \gamma'(s)\|\, ds \\ &= \frac{1}{2}\int_I \|\gamma(s) \times \gamma'(s)\|\, ds. \end{aligned}$$

(iii) From the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$ and the assumption on γ we obtain, for $s \in I$,

$$\alpha'(s) = \frac{\|\gamma(s) \times \gamma'(s)\|}{\|\gamma(s)\|^2} > 0.$$

It follows that $\alpha : I \to \mathbf{R}$ is monotonically increasing and injective, as a consequence. Therefore $\alpha : I \to\ ]\,0, \alpha(l)\,[$ is a bijection, and so is $\beta :\ ]\,0, \alpha(l)\,[\ \to I$. Observe that β is a C^1 function. Next, we verify that Υ is injective. Indeed, suppose $\Upsilon(s,t) = \Upsilon(\widetilde{s}, \widetilde{t})$. By projection onto the second factors we see that $\alpha(s) = \alpha(\widetilde{s})$, and so $s = \widetilde{s}$. But this implies $t = \widetilde{t}$ too. By definition Υ is surjective and therefore it is bijective. Now suppose $\Upsilon(s,t) = (r,\alpha) \in V$, then $s = \beta(\alpha)$ and so

$$t = \frac{r}{\|\gamma(s)\|} = \frac{r}{r(\alpha)}.$$

Since all mappings involved are of class C^1 on their domains, so is the inverse of Υ. Furthermore,

$$\upsilon = \Psi \circ \Upsilon \circ \phi^{-1} : t\gamma(s) \mapsto (s,t) \mapsto (t\|\gamma(s), \alpha(s)) \mapsto t\|\gamma(s)\|\,(\cos\alpha(s),\, \sin\alpha(s))$$

is a bijection $\mathbf{R}^3 \supset C \to \upsilon(C) \subset \mathbf{R}^2$ since all the composing maps are bijections. Obviously the mapping $t \mapsto \phi(s,t)$ has a ruling on C as its image, which then gets mapped by υ to a line segment in $\mathbf{R}^2$. Both line segments are of equal length since $\|(\cos\alpha(s),\, \sin\alpha(s))\| = 1$, for all s.

(iv) A natural parametrization of $\upsilon(C) \subset \mathbf{R}^2$ is given by

$$\psi = \upsilon \circ \phi : D \to \upsilon(C) \qquad \text{satisfying} \qquad \psi(s,t) = t\|\gamma(s)\| \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix}.$$

The evaluation of $\mathrm{area}\,(\upsilon(C))$ requires the value of $|\det D\psi(s,t)|$. Therefore we compute

$$\begin{aligned}
\frac{\partial\psi}{\partial s}(s,t) &= * \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix} + t\|\gamma(s)\|\alpha'(s) \begin{pmatrix} -\sin\alpha(s) \\ \cos\alpha(s) \end{pmatrix} \\
&= * \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix} + t\frac{\|\gamma(s)\times\gamma'(s)\|}{\|\gamma(s)\|} \begin{pmatrix} -\sin\alpha(s) \\ \cos\alpha(s) \end{pmatrix}, \\
\frac{\partial\psi}{\partial t}(s,t) &= \|\gamma(s)\| \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix},
\end{aligned}$$

where there will be no need for further specification of $*$. In fact,

$$\det D\psi(s,t) = t\|\gamma(s)\times\gamma'(s)\|| \begin{vmatrix} -\sin\alpha(s) & \cos\alpha(s) \\ \cos\alpha(s) & \sin\alpha(s) \end{vmatrix} = -t\|\gamma(s)\times\gamma'(s)\||$$

and therefore we obtain the same value as in part (ii)

$$\mathrm{area}\,(\upsilon(C)) = \int_{I\times]\,0,1\,[} t\|\gamma(s)\times\gamma'(s)\|\,d(s,t) = \frac{1}{2}\int_I \|\gamma(s)\times\gamma'(s)\|\,ds.$$

Finally, note that

$$r(\cos\alpha,\sin\alpha) = \upsilon(\gamma(s)) = \|\gamma(s)\|\,(\cos\alpha(s),\,\sin\alpha(s)) \in \upsilon(C)$$

implies $\alpha = \alpha(s)$, so $s = \beta(\alpha)$ and therefore

$$r = \|\gamma(s)\| = \|\gamma\circ\beta(\alpha)\| = r(\alpha).$$

Accoerdingly the formula in the penultimate display in Example 6.6.4 takes the form

$$\mathrm{area}\,(\upsilon(C)) = \frac{1}{2}\int_0^{\alpha(l)} r(\alpha)^2\,d\alpha.$$