Solution of Exercise 7.19.

(i) According to the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$ we have, for $x \in \,]\,0,1\,]$,

$$f'(x) = -\frac{\sqrt{1-x^2}}{x}, \qquad \text{so} \qquad \sqrt{1+f'(x)^2} = \sqrt{1+\frac{1-x^2}{x^2}} = \frac{1}{x}.$$

On the basis of Formula (7.17) the desired length then equals

$$\int_x^1 \frac{1}{x}\,dx = -\log x = \log\Big(\frac{1}{x}\Big).$$

(ii) This is immediate from $\lim_{x\downarrow 0}\log(\frac{1}{x}) = \infty$.

(iii) Consider the parametrization ϕ of the pseudosphere from Exercise 5.51.(v). The pseudosphere is a C^∞ manifold at all of its points with the exception of the points belonging to the "rim", which consists of points of the form $\phi(\frac{\pi}{2}, t)$, for $-\pi < t \leq \pi$. It follows from the formulae in that part that the outer normal to the pseudosphere at $\phi(s,t)$ is given by

$$-\psi(t,-s), \qquad \text{where} \qquad \psi(t,s) = (\cos t\cos s, \sin t\cos s, \sin s)$$

is the usual parametrization of the unit sphere S^2. As a consequence, the Euclidean density ω_ϕ associated with ϕ satisfies $\omega_\phi(s,t) = |\cos s|$, for $0 < s < \frac{\pi}{2}$ or $\frac{\pi}{2} < s < \pi$. In turns this implies that the mapping which assigns to a point of the pseudosphere not belonging to the rim its outer normal at that point (in other words, the Gauss mapping from Section 5.7) gives an area-preserving diffeomorphism between the pseudosphere minus its rim and an open dense subset of S^2. This explains why the pseudosphere has the same area 4π as the sphere. Alternatively, by mere computation, one finds for the area of the pseudosphere

$$\int_{-\pi}^{\pi}\int_0^{\pi} |\cos s|\,ds\,dt = 4\pi\int_0^{\frac{\pi}{2}} \cos s\,ds = 4\pi.$$