Solution of Exercise 7.53.

(i) The equality follows upon introduction of the new variable $\eta \in S^{n-1}$ by means of $x + r\eta = y \in S(x;r)$ in the integral defining $m_f(x,r)$. Furthermore, use $\mathrm{hyperarea}_{n-1}(S(x;r)) = r^{n-1}|S^{n-1}|$ and $d_{n-1}y = r^{n-1}\,d_{n-1}\eta$.

(ii) For every $y \in S^{n-1}$ one has $\nu(y) = y$, while the chain rule implies

$$\frac{\partial f}{\partial r}(x+ry) = Df(x+ry)\,y = \langle\,\mathrm{grad}\,f(x+ry), y\,\rangle = \langle\,\mathrm{grad}\,f(x+ry), \nu(y)\,\rangle.$$

As a consequence the desired formula follows from the identity in part (i) by means of the Differentiation Theorem 6.12.4.

(iii) As in part (ii) the formula is a direct consequence of the chain rule.

(iv) Recognize the integral in part (ii) as the right-hand side (up to a scalar) of the identity in Gauss' Divergence Theorem 7.8.5, with the role of f played by the vector field from part (iii). The formula from part (iii) taken in conjunction with Gauss' Divergence Theorem then leads to the desired identity.

(v) If f is harmonic on Ω, then the right-hand side of the identity in part (iv) vanishes and as a consequence $r \mapsto m_f(x,r)$ is a constant function, for $r > 0$ sufficiently small. From $\lim_{r\downarrow 0} m_f(x,r) = f(x)$ one deduces that f possesses the mean value property on Ω. Conversely, suppose this to be the case for f and assume f to be of class C^2. Then use the identity in (iv) once again to obtain $\int_{B^n} \Delta f(x+rx')\,dx' = 0$, for all $x \in \Omega$ and admissible $r > 0$. In view of the continuity of Δf this only is possible if $\Delta f(x) = 0$, for all $x \in \Omega$.

(vi) Let $D \subset \mathbf{R}^{n-1}$ be open and $\phi : D \to S^{n-1}$ a C^1 parametrization of an open subset of S^{n-1} having negligible complement. Introduce the open unit ball $B^n(r)$ in $\mathbf{R}^n$ of center 0 and of radius r and further

$$\Psi : \,]\,0,r\,[\,\times D \to B^n(r) \qquad \text{by} \qquad \Psi(\rho,y) = \rho\,\phi(y).$$

As in Example 7.4.12 one sees that Ψ is a diffeomorphism onto an open dense subset of $B^n(r)$ satisfying

$$|\det D\Psi(\rho,y)| = |\det(\phi(y) \mid \rho\,D\phi(y))| = \rho^{n-1}\omega_\phi(y).$$

Next use the Change of Variables Theorem 6.6.1 to deduce

$$\begin{aligned} r^n \int_{B^n} f(x+rx')\,dx' &= \int_{B^n(r)} f(x+z)\,dz = \int_0^r \rho^{n-1} \int_{S^{n-1}} f(x+\rho y)\,d_{n-1}y\,d\rho \\ &= |S^{n-1}| \int_0^r \rho^{n-1}\,m_f(x,\rho)\,d\rho. \end{aligned}$$

(vii) According to the parts (iv) and (vi) one has

$$\frac{\partial m_f}{\partial r}(x,r) = \frac{r^n}{r^{n-1}|S^{n-1}|}\,\Delta_x\Big(\int_{B^n} f(x+rx')\,dx'\Big) = \frac{1}{r^{n-1}}\,\Delta_x\Big(\int_0^r \rho^{n-1}\,m_f(x,\rho)\,d\rho\Big),$$

which implies by means of the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$

$$\begin{aligned} \frac{\partial}{\partial r}\Big(r^{n-1}\,\frac{\partial m_f}{\partial r}(x,r)\Big) &= \frac{\partial}{\partial r}\Delta_x\Big(\int_0^r \rho^{n-1}\,m_f(x,\rho)\,d\rho\Big) \\ &= \Delta_x\frac{\partial}{\partial r}\Big(\int_0^r \rho^{n-1}\,m_f(x,\rho)\,d\rho\Big) = \Delta_x(r^{n-1}\,m_f(x,r)) \\ &= r^{n-1}\Delta_x m_f(x,r). \end{aligned}$$

Darboux's equation $(\star)$ now follows from

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\,\frac{\partial}{\partial r}.$$