Solution of Exercise 7.65.

(i) Suppose $f \in F$ to be harmonic on Ω and $g \in F_0$. On the basis of $\Delta f = 0$ and $g = 0$ on $\partial\Omega$ one obtains from Green's first identity (7.65)

$$\begin{aligned} 0 &= \int_\Omega (g\,\Delta f)(x)\,dx = \int_{\partial\Omega} \Big(g\,\frac{\partial f}{\partial \nu}\Big)(y)\,d_{n-1}y - \int_\Omega \langle\,\mathrm{grad}\,f,\ \mathrm{grad}\,g\,\rangle(x)\,dx \\ &= -\int_\Omega \langle\,\mathrm{grad}\,f,\ \mathrm{grad}\,g\,\rangle(x)\,dx, \end{aligned}$$

which amounts to ($\star$). Conversely, if ($\star$) is valid for all $g \in F_0$, then again by Green's first identity

$$\int_\Omega (g\,\Delta f)(x)\,dx = 0; \qquad \text{thus} \qquad \Delta f(x) = 0 \qquad (x \in \Omega).$$

(ii) Suppose ($\star\star$) to be true and consider $f \in F_k$ and $g \in F_0$. Note that $D(t)$ as in the hint satisfies

$$D(t) = \int_\Omega \|\,\mathrm{grad}\,f(x)\|^2\,dx + 2t\int_\Omega \langle \mathrm{grad}\,f(x), \mathrm{grad}\,g(x)\rangle\,dx + t^2\int_\Omega \|\,\mathrm{grad}\,g(x)\|^2\,dx.$$

Because $D(t)$ attains its minimum at $t = 0$, one obtains

$$D'(0) = 2\int_\Omega \langle \mathrm{grad}\,f(x), \mathrm{grad}\,g(x)\rangle\,dx = 0.$$

Then f is harmonic according to part (i). The converse statement follows directly from the hint.