Exercise 0.1 (Area of surface in $\mathbf{C}^2$). As usual, we identify $z = y_1 + iy_2 \in \mathbf{C}$ with $y = (y_1, y_2) \in \mathbf{R}^2$. In particular, an open set $D \subset \mathbf{C}$ is identified with the corresponding $D \subset \mathbf{R}^2$ and a complex-differentiable function $f : D \to \mathbf{C}$ with the vector field $f = (f_1, f_2) : D \to \mathbf{R}^2$. Thus, we will study $\mathrm{graph}(f) \subset \mathbf{C}^2$ in the form of the following set:

$$V = \{\, (y, f(y)) \in \mathbf{R}^4 \mid y \in D \subset \mathbf{R}^2 \,\} = \mathrm{im}(\phi) \qquad \text{with}$$

$$\phi : D \to \mathbf{R}^4 \qquad \text{given by} \qquad \phi(y) = (y_1, y_2, f_1(y), f_2(y)).$$

It is obvious that V is a C^∞ submanifold in $\mathbf{R}^4$ of dimension 2 and that ϕ is a C^∞ embedding.

(i) Compute the Euclidean 2-dimensional density ω_ϕ on V determined by ϕ. Next, use the Cauchy–Riemann equations $D_1 f_1 = D_2 f_2$ and $D_1 f_2 = -D_2 f_1$ to show the following identity of functions on $\mathbf{R}^2$:

$$\omega_\phi = 1 + \|\,\mathrm{grad}\, f_1\|^2 = 1 + \|\,\mathrm{grad}\, f_2\|^2.$$

Suppose D to be a bounded open Jordan measurable set and deduce

$$\mathrm{vol}_2(V) = \mathrm{area}(D) + \int_D \|\,\mathrm{grad}\, f_1(y)\|^2 \, dy.$$

(ii) Suppose $D = \{\, z \in \mathbf{C} \mid |z| < 1 \,\}$ and $f(z) = z^2$. In this case verify that $\mathrm{vol}_2(V) = 3\pi$.

Solution of Exercise 0.1

(i) According to Lemma 8.3.10.(i) and (ii) the Cauchy–Riemann equations apply to the real and imaginary parts f_1 and f_2 of the holomorphic function f; consequently, we have the following equality of mappings $\mathbf{R}^2 \to \mathrm{Mat}(2, \mathbf{R})$:

$$\begin{aligned}
(D\phi)^t\, D\phi &= \begin{pmatrix} 1 & 0 & D_1 f_1 & D_1 f_2 \\ 0 & 1 & D_2 f_1 & D_2 f_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ D_1 f_1 & D_2 f_1 \\ D_1 f_2 & D_2 f_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 + (D_1 f_1)^2 + (D_1 f_2)^2 & D_1 f_1\, D_2 f_1 + D_1 f_2\, D_2 f_2 \\ D_1 f_1\, D_2 f_1 + D_1 f_2\, D_2 f_2 & 1 + (D_2 f_1)^2 + (D_2 f_2)^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 + \|\,\mathrm{grad}\, f_1\|^2 & 0 \\ 0 & 1 + \|\,\mathrm{grad}\, f_1\|^2 \end{pmatrix}.
\end{aligned}$$

Indeed, the coefficient of index $(2, 1)$ equals $D_1 f_1\, D_2 f_1 - D_2 f_1\, D_1 f_1 = 0$. In view of Definition 7.3.1. – Theorem we obtain

$$\omega_\phi = \sqrt{\det\left((D\phi)^t\, D\phi\right)} = \sqrt{(1 + \|\,\mathrm{grad}\, f_1\|^2)^2} = 1 + \|\,\mathrm{grad}\, f_1\|^2.$$

The last assertion now follows, because

$$\mathrm{vol}_2(V) = \int_V d_2 x = \int_D \omega_\phi(y)\, dy = \int_D (1 + \|\,\mathrm{grad}\, f_1(y)\|^2)\, dy.$$

(ii) $f_1(y) = \mathrm{Re}(y_1 + iy_2)^2 = y_1^2 - y_2^2$, hence $\mathrm{grad}\, f_1(y) = 2(y_1, -y_2)$ and so $\|\,\mathrm{grad}\, f_1(y)\|^2 = 4\|y\|^2$. Introducing polar coordinates (r, α) in $\mathbf{R}^2 \setminus \{\, (y_1, 0) \in \mathbf{R}^2 \mid y_1 \leq 0 \,\}$, which leads to a C^1 change of coordinates, we find

$$\int_D \|\,\mathrm{grad}\, f_1(y)\|^2 \, dy = \int_{-\pi}^{\pi} \int_0^1 4r^3 \, dr \, d\alpha = 2\pi [r^4]_0^1 = 2\pi.$$

The assertion is now a consequence from $\mathrm{area}(D) = \pi$.