**Exercise 0.1 (Averaging norms of vectors over a submanifold).** For a bounded $C^1$ submanifold $V$ in $\mathbf{R}^n$ of dimension $d$, define

$$A(V) = \frac{1}{\mathrm{vol}_d(V)} \int_V \|x\| \, d_d x.$$

Then $A(V)$ represents the average norm of a vector belonging to the submanifold $V$.

(i) Consider $B^2 = \{\, x \in \mathbf{R}^2 \mid \|x\| < 1 \,\}$ and compute $A(B^2)$.

(ii) Set $\square = \{\, x \in \mathbf{R}^2 \mid 0 < x_j < 1,\ 1 \le j \le 2 \,\}$ and show

$$A(\square) = \frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2}) = 0.765\,195\cdots.$$

**Hint**. Introduce polar coordinates $(r, \alpha)$ in $\square$. Next, one may apply without proof

$$\int \frac{1}{\cos^3\alpha}\, d\alpha = \int \frac{1}{(1-\sin^2\alpha)^2}\, d(\sin\alpha) = \frac{\sin\alpha}{2\cos^2\alpha} + \frac{1}{2}\log\tan\Big(\frac{\pi}{4} + \frac{\alpha}{2}\Big).$$

Furthermore, use that $\cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8} = \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$ implies

$$\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right. \frac{\pi}{8} = \frac{1}{2}\sqrt{2 \mp \sqrt{2}} \qquad \text{and} \qquad \left\{\begin{matrix}\sin\\ \cos\end{matrix}\right. \frac{3\pi}{8} = \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right. \Big(\frac{\pi}{2} - \frac{3\pi}{8}\Big) = \frac{1}{2}\sqrt{2 \pm \sqrt{2}}.$$

(iii) Evaluate $A(V)$ where

$$V = \mathrm{im}(\phi) \qquad \text{with} \qquad \phi : \,]-1,1[\, \to \mathbf{R}^3 \qquad \text{given by} \qquad \phi(t) = (\cos t, \sin t, t).$$

**Hint**. In the computation of the integral, set $t = \tan\alpha$.

**Background**. The value of the integral in part (ii) occurs, for instance, in the calculation of the expected distance between two random points on different sides of the unit square.

**Solution of Exercise 0.1**

(i) Using polar coordinate $(r, \alpha)$ in $B^2$ one finds

$$\int_{B^2} \|x\|\, dx = \int_{-\pi}^{\pi}\int_0^1 r^2\, dr\, d\alpha = 2\pi\frac{1}{3} = \frac{2\pi}{3}.$$

Hence $A(B^2) = \frac{2}{3}$.

(ii) Introduction of polar coordinates leads to (compare with Exercise 6.15)

$$\int_\square \|x\|\, dx = 2\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\alpha}} r^2\, dr\, d\alpha = \frac{2}{3}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\alpha}\, d\alpha.$$

Now

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\alpha}\, d\alpha = \frac{1}{2}\sqrt{2} + \frac{1}{2}\log\bigg(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\bigg) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\log\Big(\frac{2+\sqrt{2}}{\sqrt{4-2}}\Big).$$

The penultimate equality is obtained by multiplying the numerator and denominator of the argument of the log by $\sqrt{2+\sqrt{2}}$. Therefore $A(\square) = \frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$.

(iii) One obtains, for $-1 < t < 1$,

$$\|D\phi(t)\| = \|(-\sin t, \cos t, 1)\| = \sqrt{2} \qquad \text{and} \qquad \|\phi(t)\| = \sqrt{1+t^2} = \sqrt{1+\tan^2\alpha} = \frac{1}{\cos\alpha}.$$

Here $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$, because $\tan\frac{\pi}{4} = 1$. Accordingly $\frac{dt}{d\alpha} = \frac{d\tan\alpha}{d\alpha} = \frac{1}{\cos^2\alpha}$ implies

$$\int_V \|x\|\, d_1x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\phi(t)\|\sqrt{2}\, dt = 2\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\alpha}\, d\alpha = 2 + \sqrt{2}\log(1+\sqrt{2}).$$

One concludes $A(V) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2})$.