Exercise 0.1 (Catenoid). As usual, define $\cosh s = \frac{1}{2}(e^s + e^{-s})$, and furthermore

$$\phi : \mathbf{R}^2 \to \mathbf{R}^3 \qquad \text{by} \qquad \phi(s,t) = (\cosh s \cos t,\ \cosh s \sin t,\ s);$$

and consider the surface $C = \operatorname{im}(\phi)$. Without proof one may use that C is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2. (Soap films spanned by two concentric parallel circles often assume the shape of a part of this surface.) For usage in this exercise recall the formulae

$$\sinh s = \frac{1}{2}(e^s - e^{-s}), \quad \cosh^2 s - \sinh^2 s = 1, \quad \cosh^2 s + \sinh^2 s = \cosh 2s, \quad 2\cosh s \sinh s = \sinh 2s.$$

Let $a \in \mathbf{R}_+$ be arbitrarily chosen.

(i) Prove that the length of the helicoidal curve $K_a = \{\, \phi(s,s) \mid |s| \le a \,\}$ on C equals $2\sqrt{2}\sinh a$.

Define $f : \mathbf{R}_+ \to \mathbf{R}$ by $f(a) = \pi(a + \cosh a \sinh a)$.

(ii) Show that the area of the subset C_a of C consisting of the $x \in C$ with $|x_3| < a$ is given by $2f(a)$.

For $a \in \mathbf{R}_+$, define $\Omega_a \subset \mathbf{R}^3$ to be the bounded open subset bounded by C_a and the two disks

$$D_a^{\pm} = \{\, x \in \mathbf{R}^3 \mid x_1^2 + x_2^2 \le \cosh^2 a,\ x_3 = \pm a \,\}.$$

(iii) Using 3-dimensional integration verify that $\operatorname{vol}_3(\Omega_a) = f(a)$.

(iv) Apply Gauss' Divergence Theorem with the open set Ω_a and the vector field $f : \mathbf{R}^3 \to \mathbf{R}^3$ satisfying $f(x) = (x_1, x_2, 0)$, and in this way explain the relation between the results from parts (ii) en (iii).

Solution of Exercise 0.1

(i) For $s \in \mathbf{R}$ and $\psi(s) := \phi(s,s) = (\cosh s \cos s, \cosh s \sin s, s)$, we have

$$D\psi(s) = (\sinh s \cos s - \cosh s \sin s, \sinh s \sin s + \cosh s \cos s, 1),$$
$$\|D\psi(s)\|^2 = \sinh^2 s + \cosh^2 s + 1 = \sinh^2 s + \cosh^2 s + \cosh^2 s - \sinh^2 s = 2\cosh^2 s.$$

Therefore the desired length is given by

$$\sqrt{2}\int_{-a}^{a} \cosh s\, ds = 2\sqrt{2}[\sinh s]_0^a = 2\sqrt{2}\sinh a.$$

(ii) We have the following equalities of vectors in $\mathbf{R}^3$, evaluated at the point $(s,t) \in \mathbf{R}^2$,

$$\frac{\partial\phi}{\partial s} = \begin{pmatrix} \sinh s \cos t \\ \sinh s \sin t \\ 1 \end{pmatrix}, \qquad \frac{\partial\phi}{\partial t} = \begin{pmatrix} -\cosh s \sin t \\ \cosh s \cos t \\ 0 \end{pmatrix}, \qquad \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial t} = -\cosh s \begin{pmatrix} \cos t \\ \sin t \\ -\sinh s \end{pmatrix},$$

$$\text{hence} \qquad \left\| \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial t} \right\|(s,t) = \cosh s\sqrt{1 + \sinh^2 s} = \cosh^2 s.$$

Now

$$2\int_0^a \cosh^2 s\,ds = \int_0^a (1+\cosh 2s)\,ds = [s + \frac{1}{2}\sinh 2s]_0^a = a + \cosh a \sinh a.$$

Note that ϕ is not an embedding with image equal to C_a, but that we can make it so, up to a negligible set, by restricting ϕ to the subset $]-a, a[\,\times\,]-\pi, \pi\,[$ of $\mathbf{R}^2$. This gives

$$\text{area}(C_a) = 2\int_{-\pi}^{\pi}\int_0^a \left\| \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial t} \right\| (s,t)\,ds\,dt = 4\pi \int_0^a \cosh^2 s\,ds = 2f(a).$$

(iii) Introduce cylindrical coordinates $x = \Psi(r,t,s) = (r\cos t, r\sin t, s)$ in $\mathbf{R}^3$; then it is a well-known computation that $|\det D\Psi(r,t,s)| = r$. Applying the Change of Variables Theorem 6.6.1 and the computation of the definite integral of $\cosh^2$ from part (ii) we get

$$\text{vol}_3(\Omega_a) \quad = 2\int_0^a \int_{-\pi}^{\pi} \int_0^{\cosh s} r\,dr\,dt\,ds = 2\pi \int_0^a \cosh^2 s\,ds = f(a).$$

(iv) Gauss' Divergence Theorem 7.8.5 asserts

$$\int_{\Omega_a} \text{div}\, f(x)\,dx = \int_{C_a} \langle f, \nu\rangle(y)\,d_2y + \sum_{\pm} \int_{D_a^{\pm}} \langle f, \nu\rangle(y)\,d_2y.$$

Now $\text{div}\, f = 2$. Furthermore, for $y \in C_a$, we obtain from the computation of the exterior product and its norm in part (ii), inserting a minus sign because we need the outer normal,

$$\langle f, \nu\rangle(y) = \left\langle \begin{pmatrix} \cosh s \cos t \\ \cosh s \sin t \\ 0 \end{pmatrix}, \frac{1}{\cosh s} \begin{pmatrix} \cos t \\ \sin t \\ -\sinh s \end{pmatrix} \right\rangle = 1.$$

And finally $\nu(y) = \pm e_3$ implies $\langle f, \nu\rangle(y) = 0$, for all $y \in D_a^{\pm}$. As a consequence we obtain on the strength of part (ii)

$$2\,\text{vol}_3(\Omega_a) = \text{area}(C_a) = 2f(a).$$