Exercise 0.1 (Rate of change of circulation of vector field around moving curve). Write $I = [\,0,1\,]$, let $U \subset \mathbf{R}$ be open and suppose $\gamma : I^2 \to U$ is a C^1 mapping. Define the t-dependent compact curve

$$\gamma_t : I \to U \qquad \text{by} \qquad \gamma_t = \gamma(\cdot, t).$$

Let $f : U \to \mathbf{R}$ be a C^1 function. The rate of change of the integral of a function over a t-dependent curve is then given by the following formula, which is a direct consequence of the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$:

$$\frac{d}{dt}\int_{\gamma_t(0)}^{\gamma_t(1)} f(x)\,dx = f_t(\gamma_t(1))\frac{\partial\gamma_t}{\partial t}(1) - f_t(\gamma_t(0))\frac{\partial\gamma_t}{\partial t}(0) \qquad (t \in I).$$

After this introductory remark we formulate an analogous result in dimension 3.

Let $U \subset \mathbf{R}^3$ be open and suppose $\gamma : I^2 \to U$ is a C^2 mapping. Define the t-dependent compact curve

$$\gamma_t : I \to U \qquad \text{by} \qquad \gamma_t = \gamma(\cdot, t), \qquad \text{and also} \qquad v \circ \gamma(s,t) := v_t \circ \gamma_t(s) := D_2\gamma(s,t) \in \mathbf{R}^3,$$

the velocity of the point $\gamma_t(s)$ at time $t \in I$. Let $f : U \to \mathbf{R}^3$ be a C^1 vector field on U. Consider

$$\int_{\gamma_t} \langle\, f(y), d_1 y\,\rangle = \int_I \langle\, f(\gamma(s,t),t),\, D_1\gamma(s,t)\,\rangle\, ds \qquad (t \in I),$$

the circulation of the vector field f around the curve γ_t. In two steps we will prove the following formula for the rate of change of this integral:

$$\frac{d}{dt}\int_{\gamma_t} \langle\, f(y), d_1 y\,\rangle = \int_{\gamma_t} \langle\, ((\operatorname{curl} f) \times v_t)(y), d_1 y\,\rangle + \langle\, f, v_t\,\rangle \circ \gamma_t(1) - \langle\, f, v_t\,\rangle \circ \gamma_t(0) \qquad (t \in I).$$

(i) Prove by means of the chain rule the following identities of functions on I^2:

$$D_2\langle\, f \circ \gamma,\, D_1\gamma\,\rangle - D_1\langle\, f \circ \gamma,\, D_2\gamma\,\rangle = \langle\, (Af) \circ \gamma \cdot D_2\gamma,\, D_1\gamma\,\rangle = \langle\, ((\operatorname{curl} f) \times v) \circ \gamma, D_1\gamma\,\rangle.$$

Here $Af(x) = Df(x) - Df(x)^t \in \operatorname{End}(\mathbf{R}^3)$ with ^{t} denoting the adjoint linear operator with respect to the standard inner product on $\mathbf{R}^3$.

(ii) Next, verify the formula for the rate of change on the basis of interchange of differentiation and integration and of part (i).

Solution of Exercise 0.1

(i) On the basis of the chain rule we find the following identities of functions on I^2:

$$\begin{aligned} D_2\langle\, f \circ \gamma,\, D_1\gamma\,\rangle &= \langle\, (Df) \circ \gamma \cdot D_2\gamma,\, D_1\gamma\,\rangle \; + \langle\, f \circ \gamma,\, D_2 D_1\gamma\,\rangle, \\ D_1\langle\, f \circ \gamma,\, D_2\gamma\,\rangle &= \langle\, (Df)^t \circ \gamma \cdot D_2\gamma,\, D_1\gamma\,\rangle + \langle\, f \circ \gamma,\, D_1 D_2\gamma\,\rangle. \end{aligned}$$

One has, by Theorem 2.7.2, that $D_2 D_1\gamma = D_1 D_2\gamma$. Successively applying subtraction, Definition 8.1.4 and Corollary 8.1.10 we see

$$\begin{aligned} D_2\langle\, f \circ \gamma,\, D_1\gamma\,\rangle - D_1\langle\, f \circ \gamma,\, D_2\gamma\,\rangle &= \langle\, (Af) \circ \gamma \cdot D_2\gamma,\, D_1\gamma\,\rangle = \langle\, (Af \cdot v) \circ \gamma,\, D_1\gamma\,\rangle \\ &= \langle\, ((\operatorname{curl} f) \times v) \circ \gamma,\, D_1\gamma\,\rangle. \end{aligned}$$

(ii) In view of Theorem 2.10.4 we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\gamma_t}\langle\, f(y), d_1 y\,\rangle &= \int_{\gamma_t} D_2\langle\, f(y), d_1 y\,\rangle = \int_I D_2\langle\, f\circ\gamma,\ D_1\gamma\,\rangle(s,t)\,ds \\
&= \int_I \langle\,((\operatorname{curl} f)\times v)\circ\gamma,\ D_1\gamma\,\rangle(s,t)\,ds + \int_I D_1\langle\, f\circ\gamma,\ D_2\gamma\,\rangle(s,t)\,ds \\
&= \int_{\gamma_t}\langle\,((\operatorname{curl} f)\times v_t)(y), d_1 y\,\rangle + \big[\langle\, f, v_t\,\rangle\circ\gamma_t\big]_0^1.
\end{aligned}
$$

For the last equality we used the definition of v_t and the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$.